A Method for Deciding the Finiteness of Synchronous, Tabled Picture Languages

Bianca Truthe

truthe@iws.cs.uni-magdeburg.de

Fakultät für Informatik, Otto-von-Guericke-Universität Magdeburg

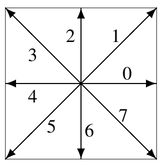
PSF 4120; D-39016 Magdeburg; Germany

Abstract

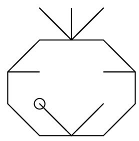
In the present paper, synchronous, tabled chain code picture systems based on Lindenmayer systems (*sT0L* systems) are studied with respect to the finiteness of their picture languages. The finiteness is proved to be decidable. Additionally, a method is given for deciding whether an *sT0L* system generates a finite picture language or not.

1. Introduction

Chain code picture systems give a possibility for describing pictures. They are based on generating words over a special alphabet and interpreting these words as pictures. They can be regarded as a formal description of the method of working of plotters. H. FREEMAN introduced chain code picture languages ([Fr61]). A picture is formed by a sequence of drawing commands represented by symbols (letters). A string describes a picture, which is built by the drawing commands of its letters. FREEMAN uses the alphabet $\{0, \ldots, 7\}$, whose elements are interpreted according to the following sketch.



The picture to the right, for example, is generated by the word 70124034372615450467O1:
(For reconstructing begin at the circle.)



For language theoretical investigations, only the four directions 0, 2, 4, 6 are considered [DH89]. According to plotter commands, r, u, l, d are written as the directions *right*, *up*, *left*, *down*. The connection of strings and pictures suggests to search for relations between formal languages and picture sets. The first paper in that topic is [Fe68]. Chomsky-based chain code picture languages have been investigated since the 1980s ([MRW82], [SW85]).

Lindenmayer systems are biologically motivated. However, hardly any theoretical investigations on Lindenmayer-based chain code picture languages have been carried out so far [DHr92].

Context-free Lindenmayer systems are divided up in *D0L* systems (deterministic rewriting of letters), *0L* systems (rewriting non-deterministically), *DT0L* systems (selecting a replacement table non-deterministically, rewriting deterministically), and *T0L* systems (selecting a replacement table non-deterministically, rewriting non-deterministically), [RS80].

In [DHr92], synchronous chain code picture systems based on Lindenmayer systems are introduced (*sD0L*, *s0L*, *sDT0L*, *sT0L*). In [T02], a hierarchy of abstractions is developed such that the interpretation of a word as a picture passes through a multilevel process. On the basis of that hierarchy, the finiteness of *sD0L* systems, *s0L* systems and *sDT0L* systems is shown to be decidable ([T02], [T03] and [T04] resp.).

In the present paper, the decidability of the finiteness of *sT0L* systems is investigated and proved.

2. Fundamentals

The finiteness investigations on picture languages of *sT0L* systems in this paper are based on the hierarchy of abstractions developed in [T02]. In this section, the fundamental notations are gathered.

2.1. Structures over an Alphabet

Let $\mathcal{A} = \{\, r, l, u, d \,\}$ be an alphabet and $(\mathcal{A}^*, \cdot)$ be the free structure over $\mathcal{A}$ with the operation of concatenation. The elements of $\mathcal{A}^*$ are words; λ is the empty word. Let $\mathcal{A}^+$ be the set of all words of $\mathcal{A}^*$ except λ and $\mathbb{A}$ be the set of all finite, non-empty subsets of $\mathcal{A}^*$. The concatenation of two word sets $U, V \in \mathbb{A}$ produces the set of all words uv with $\mathsf{u} \in U$ and $\mathsf{v} \in V$. The set $\mathbb{A}$ with the operations $\cup$ (union) and $\cdot$ (concatenation) forms a semiring $(\mathbb{A}, \cup, \cdot)$ because $(\mathbb{A}, \cup)$ and $(\mathbb{A}, \cdot)$ are semigroups and the distributive laws are valid.

All words w of the length $|\mathsf{w}| = n$ form the set $\mathcal{A}^n$. A word $\mathsf{w} \in \mathcal{A}^n$ is composed of letters $w_1, \ldots, w_n$ unless stated otherwise: $\mathsf{w} = w_1 \cdots w_n$. In this context, $\overrightarrow{w_i}$ is the word $\overrightarrow{w_i} = w_1 \cdots w_i$ ($0 \leq i \leq n$, $\overrightarrow{w_0} = \lambda$). The number $|\mathsf{w}|_x$ is the number of occurrences of the letter x in the word w. The set of all letters occurring in w is shortly written as $[\mathsf{w}]$:

$$[\mathsf{w}] = \{\, x \mid |\mathsf{w}|_x \geq 1 \,\}.$$

The elements of $\mathcal{A}^*$ can be interpreted as mappings on $\mathbb{Z}^2$. The empty word corresponds to the identity mapping. The atomic mappings r, l, u, d assign, to a point $\mathsf{q} \in \mathbb{Z}^2$, its neighbours:

$$\begin{array}{rclcrcl} r(\mathsf{q}) & = & \mathsf{q} + (1,0), & \qquad & l(\mathsf{q}) & = & \mathsf{q} - (1,0), \\ u(\mathsf{q}) & = & \mathsf{q} + (0,1), & \qquad & d(\mathsf{q}) & = & \mathsf{q} - (0,1). \end{array}$$

The function names r, l, u, d are taken from the directions *right*, *left*, *up*, *down*. A compound word $\mathsf{vw} \in \mathcal{A}^*$ stands for the concatenated mapping $\mathsf{v} \circ \mathsf{w}$:

$$\mathsf{v} \circ \mathsf{w} : \mathbb{Z}^2 \to \mathbb{Z}^2 \quad \text{with } \mathfrak{q} \mapsto \mathsf{w}(\mathsf{v}(\mathfrak{q})).$$

The zero point of $\mathbb{Z}^2$ is denoted by $\mathfrak{o} = (0,0)$. The translations $x(\mathfrak{q}) - \mathfrak{q}$ ($x \in \mathcal{A}$) of any point $\mathfrak{q} \in \mathbb{Z}^2$ to its neighbours $x(\mathfrak{q})$ are denoted by $\mathfrak{v}_x \in \mathbb{Z}^2$. Consequently, $\mathfrak{v}_r = (1,0)$, $\mathfrak{v}_l = -(1,0)$, $\mathfrak{v}_u = (0,1)$, $\mathfrak{v}_d = -(0,1)$.

The interpretation of words as mappings on $\mathbb{Z}^2$ is a homomorphism from the free structure $(\mathcal{A}, \cdot)$ into the free structure $(\mathcal{A}, \circ)$. The operator $\circ$ does not need to be written if the context shows which operation is meant.

The mappings r and l as well as u and d are inverse to each other. The mappings ru and ur as well as ld and dl assign, to a point $\mathfrak{q}$, its diagonal neighbours:

$$ru(\mathfrak{q}) = ur(\mathfrak{q}) = \mathfrak{q} + (1,1), \quad ld(\mathfrak{q}) = dl(\mathfrak{q}) = \mathfrak{q} - (1,1).$$

The mapping which leads, together with a mapping x, to a diagonal neighbour is denoted by $x^\perp$. The inverse mappings of two mappings x and $x^\perp$ are denoted by $\bar{x}$ and $\bar{x}^\perp$ respectively. The table to the right shows the corresponding mappings.

x	$\bar{x}$	$x^\perp$	$\bar{x}^\perp$
r	l	u	d
l	r	d	u
u	d	r	l
d	u	l	r

2.2. Graphical Embedding

A lattice graph is a graph with the following properties: the vertex set is a subset of $\mathbb{Z}^2$, and each edge is incident to two neighbours $\mathfrak{q}$ and $x(\mathfrak{q})$ with $\mathfrak{q} \in \mathbb{Z}^2$ and $x \in \{r, l, u, d\}$. The 'position' of the vertices is essential: in general, renaming of vertices does not lead to an isomorphic lattice graph. Properties like directed, undirected, simple remain unchanged.

In [T02], we define functions that assign to each word $\mathsf{w} \in \mathcal{A}^n$

- the vertex set $\odot^{\mathfrak{a}}(\mathsf{w}) = \{ \overrightarrow{w_i}(\mathfrak{a}) \mid i = 0, \dots n \}$,
- the directed lattice graph (possibly with multiple edges),
- the simple, directed lattice graph $s^{\mathfrak{a}}(\mathsf{w})$ of $g^{\mathfrak{a}}(\mathsf{w})$ (without multiple edges),
- the edge set $\|^{\mathfrak{a}}\mathsf{w} = \{ (\overrightarrow{w_{i-1}}(\mathfrak{a}), w_i) \mid i = 1, \dots, n \}$ of $s^{\mathfrak{a}}(\mathsf{w})$, where an edge is described by a pair of a start point and a direction rather than a pair of start and end points,
- the picture (the shadow of $s^{\mathfrak{a}}(\mathsf{w})$)

$$p^{\mathfrak{a}}(\mathsf{w}) = (\odot^{\mathfrak{a}}(\mathsf{w}), \{ (\overrightarrow{w_{i-1}}(\mathfrak{a}), \overrightarrow{w_i}(\mathfrak{a})), (\overrightarrow{w_i}(\mathfrak{a}), \overrightarrow{w_{i-1}}(\mathfrak{a})) \mid i = 1, \dots, n \})$$

- and the picture area

$$\boxdot^{\mathfrak{a}}(\mathsf{w}) = \left\{ (x,y) \;\middle|\; \begin{array}{l} \underline{x}^{\mathfrak{a}}(\mathsf{w}) \le x \le \overline{x}^{\mathfrak{a}}(\mathsf{w}) \text{ and} \\ \underline{y}^{\mathfrak{a}}(\mathsf{w}) \le y \le \overline{y}^{\mathfrak{a}}(\mathsf{w}) \end{array} \right\},$$

$\underline{x}^{\mathfrak{a}}(\mathsf{w})$, $\underline{y}^{\mathfrak{a}}(\mathsf{w})$, $\overline{x}^{\mathfrak{a}}(\mathsf{w})$, and $\overline{y}^{\mathfrak{a}}(\mathsf{w})$ being the 'border coordinates' of the vertex set.

The upper index will be omitted if the mappings relate to the zero point ($\mathfrak{a} = \mathfrak{o}$).

The picture areas are rectangle sets. A rectangle set $\mathfrak{P}$ is determined by two points, the 'lower left corner' $(\underline{x}, \underline{y})$ and the 'upper right corner' $(\overline{x}, \overline{y})$ or the 'upper left corner' $(\underline{x}, \overline{y})$ and the 'lower right corner' $(\overline{x}, \underline{y})$. The notation for a rectangle set is $[(\underline{x}, \underline{y}), (\overline{x}, \overline{y})]$. The following descriptions are equivalent:

$$[(\underline{x}, \underline{y}), (\overline{x}, \overline{y})],\ [(\overline{x}, \overline{y}), (\underline{x}, \underline{y})],\ [(\underline{x}, \overline{y}), (\overline{x}, \underline{y})],\ [(\overline{x}, \underline{y}), (\underline{x}, \overline{y})].$$

Scaling a picture area $\mathfrak{P} = [\mathfrak{p}, \mathfrak{q}]$ by a factor s produces the picture area $s\mathfrak{P} = [s\mathfrak{p}, s\mathfrak{q}]$.

The union of two picture areas is not a rectangle set in general. An extended union of two picture areas shall be the picture area of the union: $\mathfrak{P}_X \Cup \mathfrak{P}_Y = \mathfrak{P}_{X \cup Y}$.

The sets $\mathcal{A}^*$ of words, $\{\, g^{\mathfrak{a}}(\mathsf{w}) \,\}$ of directed lattice graphs, $\{\, s^{\mathfrak{a}}(\mathsf{w}) \,\}$ of simple, directed lattice graphs, and $\{\, p^{\mathfrak{a}}(\mathsf{w}) \,\}$ of pictures ($\mathsf{w} \in \mathcal{A}^*, \mathfrak{a} \in \mathbb{Z}^2$) form different levels of a hierarchy of abstractions. On the lowest level, words of a language are considered. Interpreting those words as directed lattice graphs leads to the next level. One gets to the third level by reducing multiple edges to one. The highest level is reached by omitting the directions of edges.

2.3. Special Endomorphisms

Let κ, μ be two natural numbers, $\kappa, \mu \in \mathbb{N}_0$. An endomorphism h on the semiring $(\mathbb{A}, \cup, \cdot)$ is called a (κ, μ)-endomorphism if, for all $x \in \mathcal{A}$, the following conditions (called synchronization conditions) are satisfied: If $x' \in h(\{\, x \,\})$, then

1. $x'(\mathfrak{o}) = \kappa \mathfrak{v}_x$ and
2. $\boxdot(x') \subseteq \kappa[\mathfrak{o}, \mathfrak{v}_x] \Cup \mu[\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$.

Applying h to a set of words W is called deriving; the set $h(W)$ is obtained in one derivation step. The n-ary composition of a (κ, μ)-endomorphism h is shortly written as h^n. Every element of $h^n(\{\, \mathsf{w} \,\})$, $\mathsf{w} \in \mathcal{A}^*$, is called an n-th derivative of w.

The following example illustrates the synchronization conditions.

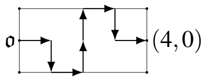
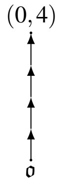
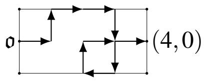
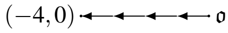
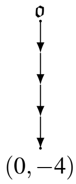
2.1. Example: Let h be an endomorphism on $(\mathbb{A}, \cup, \cdot)$ with

$$h(\{\, r \,\}) = \{\, rdruurdr, rurrddlurr \,\} \text{ and } h(\{\, x \,\}) = \{\, xxxx \,\} \text{ for } x \neq r.$$

The first synchronization condition is satisfied for $\kappa = 4$:

$$\begin{aligned}
(rdruurdr)(\mathfrak{o}) &= \mathfrak{v}_r + \mathfrak{v}_d + \mathfrak{v}_r + \mathfrak{v}_u + \mathfrak{v}_u + \mathfrak{v}_r + \mathfrak{v}_d + \mathfrak{v}_r = 4\mathfrak{v}_r,\\
(rurrddlurr)(\mathfrak{o}) &= \mathfrak{v}_r + \mathfrak{v}_u + \mathfrak{v}_r + \mathfrak{v}_r + \mathfrak{v}_d + \mathfrak{v}_d + \mathfrak{v}_l + \mathfrak{v}_u + \mathfrak{v}_r + \mathfrak{v}_r = 4\mathfrak{v}_r,\\
(xxxx)(\mathfrak{o}) &= 4\mathfrak{v}_x.
\end{aligned}$$

The simple, directed lattice graphs and their picture areas are shown below.

Derivative	Corresponding lattice graph	Derivative	Corresponding lattice graph
$rdruurdr$		$uuuu$	
$rurrddlurr$			
$llll$		$dddd$	

All points of a derivative $x' \in h(\{x\})$ lie in the picture area $4[\mathfrak{o}, \mathfrak{v}_x] \Cup [\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$. Hence, the endomorphism h is a $(4,1)$-endomorphism.
The first synchronization condition is about the position of the endpoints of the pictures belonging to the first derivatives of the alphabet; the second condition forces every picture of a derivative of $x \in \mathcal{A}$ to lie in a certain rectangle.

Let h be a (κ, μ)-endomorphism, w be a word of $\mathcal{A}^n$ and $W \in \mathbb{A}$ a finite word set. Then one has

$$h(\{\mathsf{w}\}) = h(\{w_1 \cdots w_n\}) = h(\{w_1\} \cdots \{w_n\}) = h(\{w_1\}) \cdots h(\{w_n\}),$$
$$h(W) = \bigcup_{\mathsf{w} \in W} h(\{\mathsf{w}\}).$$

The cardinality of a set M is denoted by $|M|$; consequently, $|h(\{\mathsf{w}\})|$ ist the number of the first derivatives of w.

Let $\mu_0 \in \mathbb{N}_0$ be a natural number. For any natural number $\mu \geq \mu_0$, the picture area $\mu_0[\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$ is a subset of $\mu[\mathfrak{v}_{x^\perp}, \mathfrak{v}_{\bar{x}^\perp}]$.

2.2. Proposition: *Each (κ, μ)-endomorphism is also a $(\kappa, \mu+1)$-endomorphism.*

The parameter κ is a factor of the length changing in one derivation step; in the case of $\kappa = 0$, the endomorphism is called length contracting, in the case of $\kappa = 1$, length constant and in the case $\kappa > 1$, length expanding. The parameter μ is an upper bound of the width changing in one derivation step.

If all atomic values of a (κ, μ)-endomorphism h have only one element, then each word has exactly one derivative; the set signs are omitted in that case: $h(\mathsf{w}) = \mathsf{w}'$.

2.4. Chain Code Picture Systems

In this section, synchronous, tabled, context-free chain code picture systems based on Lindenmayer systems (*sT0L* systems) are defined and set in connection with simple non-deterministic systems (*s0L* systems).

2.4.1. *sT0L* Systems

An *sT0L* system is a triple $G = (\mathcal{A}, h, \omega)$ with the alphabet $\mathcal{A} = \{ r, l, u, d \}$, a finite, non-empty set $h = \{ h_1, \ldots, h_m \}$ of (κ_i, μ_i)-endomorphisms h_i, and a non-empty word ω over $\mathcal{A}$ (called the axiom).

The set of all n-ary compositions of elements of h is denoted by h^n

$$h^n = \{ h_{i_1} \circ \cdots \circ h_{i_n} \mid i_j \in \{ 1, \ldots, m \};\ j = 1, \ldots, n \};$$

applying all those compositions to a set of words W yields the set

$$h^n(W) = \left\{ h^{(n)}(\mathsf{w}) \;\middle|\; h^{(n)} \in h^n, \mathsf{w} \in W \right\}.$$

Composing all elements of h^n with an (κ, μ)-endomorphism g yields

$$h^n \circ g = \left\{ h^{(n)} \circ g \;\middle|\; h^{(n)} \in h^n \right\},$$

similiary, the composition of two sets h^k and h^n is

$$h^k \circ h^n = \left\{ h^{(k)} \circ h^{(n)} \;\middle|\; h^{(k)} \in h^k, h^{(n)} \in h^n \right\} = h^{k+n}.$$

The picture language P_G generated by an *sT0L* system G is defined to be the set of all pictures of derivatives of the axiom ω:

$$P_G = \{ p(\mathsf{w}) \mid \mathsf{w} \in h^n(\{ \omega \}), n \in \mathbb{N}_0 \}.$$

An *sT0L* system is called length expanding if at least one (κ_i, μ_i)-endomorphism $h_i \in h$ has this property. An *sT0L* system is called length contracting if all endomorphisms of h are length contracting. In the other cases, at least one endomorphism of h is length constant while the others are length contracting. Those *sT0L* systems are called length constant.

An *s0L* system is an *sT0L* system with only one endomorphism ([T03]). In order to use results about simple non-deterministic systems (*s0L* systems), simple non-deterministic subsystems and supersystems are defined below.

An *s0L* system $U = (\mathcal{A}, h_\circ, \omega)$ is called a simple non-deterministic subsystem of a tabled chain code picture system $G = (\mathcal{A}, h, \omega)$ (written $U \sqsubseteq G$) if each derivative of any $\mathsf{w} \in \mathcal{A}^*$ with respect to U is also a derivative with respect to G. An *s0L* subsystem $U = (\mathcal{A}, h_\circ, \omega)$ of an *sT0L* system $G = (\mathcal{A}, \{ h_1, \ldots, h_m \}, \omega)$ is said to be maximal if $h_\circ(\{ \mathsf{w} \}) = h_i(\{ \mathsf{w} \})$ for an $i \in \{ 1, \ldots, m \}$. An *s0L* subsystem $U_i = (\mathcal{A}, h_i, \omega)$ of an *sT0L* system generates the picture language

$$P_{U_i} = \{ p(\mathsf{w}) \mid \mathsf{w} \in h_i^n(\{ \omega \}), n \in \mathbb{N}_0 \}.$$

2.3. Proposition: *The picture language generated by an s0L subsystem of an sT0L system G is a subset of the picture language generated by G.*

An *s0L* system $S = (\mathcal{A}, h_\circ, \omega)$ is called a simple non-deterministic supersystem (shortly *s0L* supersystem) of a tabled chain code picture system G (written $S \sqsupseteq G$) if each derivative of a word $\mathsf{w} \in \mathcal{A}^*$ with respect to G is also a derivative with respect to S. An *s0L* supersystem $S = (\mathcal{A}, h_\circ, \omega)$ of an *sT0L* system $G = (\mathcal{A}, \{h_1, \ldots, h_m\}, \omega)$ is said to be minimal if

$$h_\circ(\{x\}) = \bigcup_{i=1}^{m} h_i(\{x\}) \quad \text{for } x \in \mathcal{A}.$$

The following lemma concerns the existence of *s0L* supersystems.

2.4. Lemma: *Let $G = (\mathcal{A}, \{h_1, \ldots, h_m\}, \omega)$ be an sT0L system.*

1. *The system G has an s0L supersystem if and only if two natural numbers κ, μ exist such that every h_i is also an (κ, μ)-endomorphism.*
2. *If G has an s0L supersystem, it has exactly one minimal s0L supersystem.*

This lemma yields the following proposition.

2.5. Proposition: *In the case that an sT0L system G has an s0L supersystem, the picture language P_G generated by G is a subset of the picture language P_S generated by the minimal s0L supersystem S of G.*

3. Finiteness of *sT0L* systems

In this section, conditions are derived for deciding whether an *sT0L* system generates a finite picture language or not. Let $G = (\mathcal{A}, h, \omega)$ be an *sT0L* system with a set $h = \{h_1, \ldots, h_m\}$ of (κ_i, μ_i)-endomorphisms h_i. The derivative of an letter $x \in \mathcal{A}$ by a n-ary composition $h_{i_1} \circ \cdots \circ h_{i_n}$ $(h_{i_1}, \ldots, h_{i_j} \in h)$ maps the zero point $\mathfrak{o}$ to the point $\kappa_{i_1} \cdots \kappa_{i_n} \mathfrak{v}_x$. This can be concluded from the first synchronization condition by induction over n. Furthermore, for a derivative $x' \in h_i(\{x\})$ by an endomorphism $h_i \in h$, one concludes $x'(\mathfrak{o}) = \kappa_i \mathfrak{v}_x + c(\mathfrak{v}_x + \mathfrak{v}_{\bar{x}}) + d(\mathfrak{v}_{x^\perp} + \mathfrak{v}_{\bar{x}^\perp})$ for two natural numbers c, d. This means that $x^\perp$ and $\bar{x}^\perp$ have the same number of occurrences in the derivative x', and the difference of the numbers of occurrences of x and $\bar{x}$ is κ_i. These observations are summarized in the following proposition.

3.1. Proposition: *Let $x \in \mathcal{A}$. If x' is a derivative of x by an endomorphism $h_i \in h$ and $x^{(n)} \in (h_{i_1} \circ \cdots \circ h_{i_n})(\{x\})$ is an n-th derivative of x, then*

1. $x^{(n)}(\mathfrak{o}) = \kappa_{i_1} \cdots \kappa_{i_n} \mathfrak{v}_x$,
2. $|x'|_x = \kappa_i + |x'|_{\bar{x}}$ *and* $|x'|_{x^\perp} = |x'|_{\bar{x}^\perp}$.

The first statement can be extended to words.

3.2. Proposition: *Let* $\mathsf{w} \in \mathcal{A}^*$. *If* $\mathsf{w}^{(n)} \in (h_{i_1} \circ \cdots \circ h_{i_n})(\{\mathsf{w}\})$ *is an n-th derivative of* w, *then* $\mathsf{w}^{(n)}(\mathfrak{o}) = \kappa_{i_1} \cdots \kappa_{i_n} \mathsf{w}(\mathfrak{o})$.

The considered chain code picture systems are separated by their ‘length behaviour’.

3.1. Length Contracting Systems

Let $G = (\mathcal{A}, h, \omega)$ be a length contracting *sT0L* system with a finite and non-empty set $h = \{ h_1, \ldots, h_m \}$ of $(0, \mu_i)$-endomorphisms h_i. Let μ be the maximum of all μ_i. According to Proposition 2.2, every h_i is also a $(0, \mu)$-endomorphism.

The minimal *s0L* supersystem $S = (\mathcal{A}, h_\circ, \omega)$ of G is a length contracting *s0L* system and its endomorphism $h_\circ$ is also a $(0, \mu)$-endomorphism. According to [T03], the picture language P_S contains $(\mu + 1)^4 + 1$ elements at most. Following Lemma 2.4, also P_G has $(\mu + 1)^4 + 1$ elements at most.

3.3. Theorem: *Every length contracting sT0L system $G = (\mathcal{A}, \{ h_1, \ldots, h_m \}, \omega)$ with $(0, \mu)$-endomorphisms h_i $(i = 1, \ldots, m)$ generates a finite picture language P_G with $(\mu + 1)^4 + 1$ elements at most: $|P_G| \leq (\mu + 1)^4 + 1 < \infty$.*

3.2. Length Expanding Systems

Let $G = (\mathcal{A}, h, \omega)$ be a length expanding *sT0L* system with $h = \{ h_1, \ldots, h_m \}$. Then, an endomorphism h_i and the *s0L* subsystem $U = (\mathcal{A}, h_i, \omega)$ are length expanding. The picture language of any length expanding *s0L* system is infinite ([T03]). Since the picture language generated by U is a subset of the picture language generated by G (Prop. 2.3), also the picture language of G is infinite.

3.4. Theorem: *A length expanding sT0L system generates an infinite picture language.*

3.3. Length Constant Systems

Length constant *s0L* systems and length constant *sDT0L* systems can generate finite and infinite picture languages ([T03, T04]). They are special length constant *sT0L* systems. Hence, also length constant *sT0L* systems can generate finite and infinite picture languages.

In this section, a finiteness criterion for *sT0L* systems is given and proved. It is shown how one can decide whether an *sT0L* system generates a finite picture language or not.

Let $g = \{ g_1, \ldots, g_k \}$ be a finite and non-empty set of $(0, \mu)$-endomorphisms and $f = \{ f_1, \ldots, f_m \}$ be a finite, non-empty set of $(1, \mu)$-endomorphisms. The union of both sets is denoted by $h = g \cup f$. Furthermore, let $G = (\mathcal{A}, h, \omega)$ be an *sT0L* system. The investigations on deterministic, simple non-deterministic and deterministic tabled systems ([T02], [T03], [T04]) suggest the following supposition: The picture language P_G generated by G is finite if and only if, for every letter $x \in [h^2(\{ \omega \})]$ occurring in a second derivative of ω, all derivatives of x by the length constant endomorphisms of f up to the third derivation produce no other x-edge than $(\mathfrak{o}, x)$:

$$|P_G| < \infty \Longleftrightarrow \forall x \in [h^2(\{ \omega \})] : \|_x x = \|_x f(\{ x \}) = \|_x f^2(\{ x \}) = \|_x f^3(\{ x \}).$$

Suppose, a letter $x \in [h^2(\{ \omega \})]$ occurs in a second derivative of ω and $l \in \{ 1, 2, 3 \}$ is a derivation step such that an l-th derivative of x by length constant endomorphisms produces a new x-edge. Then, the edge $(\mathfrak{o}, x)$ yields an edge $(\mathfrak{q}, x)$ with $\mathfrak{q} \neq \mathfrak{o}$ after l

derivation steps, and also $(\mathfrak{o},x)$ yields the edge $(n\mathfrak{q},x)$ after nl derivation steps. Hence, infinitely many x-edges are generated. Since any picture has only finitely many edges, the picture language generated is infinite. By contraposition one part of the equivalence supposed above is proved.

3.5. Lemma: *If the picture language P_G generated by G is finite, for every letter $x \in [h^2(\{\,\omega\,\})]$ occurring in a second derivative of ω, all derivatives of x by the length constant endomorphisms of f up to the third derivation produce no other x-edge than $(\mathfrak{o},x)$:*

$$|P_G| < \infty \Longrightarrow \forall x \in [h^2(\{\,\omega\,\})]:\ \|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\}).$$

Now suppose $\|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\})$ for all letters x in a second derivative of the axiom. By case distinction about the letters produced in one derivation step, one can show the following lemma.

3.6. Lemma: *Any letter produced by G also occurs in a second derivative:*

$$[h^n(\{\,\omega\,\})] \subseteq [h^2(\{\,\omega\,\})].$$

Thus, we have $\|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\})$ for all letters x occurring in an arbitrary derivative and not only in a second derivative.

Further can be shown, how the edge set of the derivatives of a word w by the length constant endomorphisms is composed of the edge sets of the derivatives of the letters of earlier derivatives of w.

3.7. Proposition: *The edge set $\|f^n(\{\,\mathsf{w}\,\})$ of the n-th derivatives of a word $\mathsf{w} \in \mathcal{A}^*$ by the length constant endomorphisms is*

$$\|f^n(\{\,\mathsf{w}\,\}) = \bigcup_{(\mathfrak{q},x)\in\|f^{n-i}(\{\,\mathsf{w}\,\})} \|^{\mathfrak{q}} f^i(\{\,x\,\}) \quad \textit{for } i = 1,\ldots,n.$$

The edge set K_n belonging to the n-th derivation step is obtained from the edge set K_j of a lower step $j < n$ by replacing every edge of K_j by all its $(n-j)$-th derivatives.

Let $\mathsf{w} \in \mathcal{A}^*$ be a word with only those letters occurring in a second derivative of the axiom and $M_i(\mathsf{w})$ be the set of edges that can be edges of an i-th derivative of w by the length constant endomorphisms. If $\mathsf{w} = w_1 \cdots w_l$, then

$$M_i(\mathsf{w}) = M_i(w_1) \cup M_i^{\overrightarrow{w_1}(\mathfrak{o})}(w_2) \cup \cdots \cup M_i^{\overrightarrow{w_{l-1}}(\mathfrak{o})}(w_l),$$

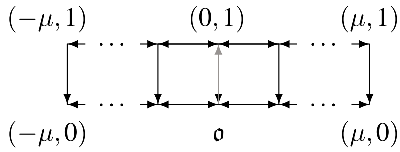
where $M_i^{\mathfrak{a}}(x)$ is obtained from $M_i(x)$ by translating the edges by $\mathfrak{a}$. For $i = 0$ and a letter x, one has $M_i(x) = M_0(x) = \|x = \{\,(\mathfrak{o},x)\,\}$. The second synchronization condition for a derivative x' of x by a $(1,\mu)$-endomorphism of f yields, for the vertex set $\odot x'$,

$$\odot x' \subseteq \left\{\, \alpha\mathfrak{v}_{x^\perp}, \alpha\mathfrak{v}_{x^\perp} + \mathfrak{v}_x \mid \alpha \in \mathbb{Z} \text{ and } -\mu \leqq \alpha \leqq \mu \,\right\}.$$

Since x does not produce another x-edge, $M_1(x)$ is

$$M_1(x) = \left\{ \begin{array}{l} (\alpha\mathfrak{v}_{x^\perp}, x^\perp), (\alpha\mathfrak{v}_{x^\perp} + \mathfrak{v}_x, x^\perp), \\ (\alpha\mathfrak{v}_{\bar{x}^\perp}, \bar{x}^\perp), (\alpha\mathfrak{v}_{\bar{x}^\perp} + \mathfrak{v}_x, \bar{x}^\perp) \end{array} \;\middle|\; \alpha \in \mathbb{Z} \text{ and } -\mu \leqq \alpha \leqq \mu - 1 \right\}$$
$$\cup \left\{ (\alpha\mathfrak{v}_{x^\perp} + \mathfrak{v}_x, \bar{x}) \mid \alpha \in \mathbb{Z} \text{ and } -\mu \leqq \alpha \leqq \mu \right\} \cup \left\{ (\mathfrak{o}, x) \right\}.$$

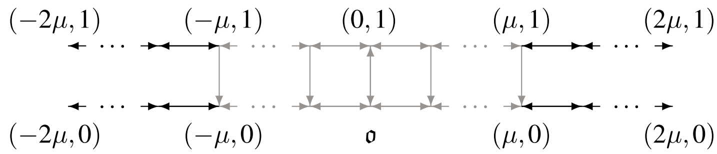
The figure shows the edges of $M_1(u)$:



The edges of $M_1(x)$ do not produce another x-edge when derived. Similarly, no new $\bar{x}$-edge is produced, because new $\bar{x}$- and x-edges arise pairwise. Thus, only new $x^\perp$- or $\bar{x}^\perp$-edges occur. They are derived from $\bar{x}$-edges of $M_1(x)$. Hence, the set $M_2(x)$ is

$$M_2(x) = \left\{ \begin{array}{l} (\alpha\mathfrak{v}_{x^\perp}, x^\perp), (\alpha\mathfrak{v}_{x^\perp} + \mathfrak{v}_x, x^\perp), \\ (\alpha\mathfrak{v}_{\bar{x}^\perp}, \bar{x}^\perp), (\alpha\mathfrak{v}_{\bar{x}^\perp} + \mathfrak{v}_x, \bar{x}^\perp) \end{array} \;\middle|\; \alpha \in \mathbb{Z} \text{ and } -2\mu \leqq \alpha \leqq 2\mu - 1 \right\}$$
$$\cup \left\{ (\alpha\mathfrak{v}_{x^\perp} + \mathfrak{v}_x, \bar{x}) \mid \alpha \in \mathbb{Z} \text{ and } -\mu \leqq \alpha \leqq \mu \right\} \cup \left\{ (\mathfrak{o}, x) \right\}.$$

The following figure shows the edges of $M_2(u)$:



If there is an edge in $M_3(x)$ but not in $M_2(x)$, this edge is produced by an edge of $M_2(x)$ that is not in $M_1(x)$. Since the $x^\perp$- and $\bar{x}^\perp$-edges occurring in $M_2(x)$ for the first time are obtained from $\bar{x}$-edges by deriving once and from the x-edge by deriving twice, they do not yield new $\bar{x}$- nor new x-edges. Hence, they do not produce new $x^\perp$- or $\bar{x}^\perp$-edges either. So, $M_3(x)$ is equal to $M_2(x)$. Since there is no edge in $M_3(x)$ that is not in $M_2(x)$, no new edges arise later: For any derivation step $n \geq 2$, the sets $M_n(x)$ and $M_2(x)$ coincide. Since $M_0(x)$ and $M_1(x)$ are included in $M_2(x)$, one obtains the following inclusion for all derivation steps $n \in \mathbb{N}_0$:

$$M_n(x) \subseteq M_2(x).$$

Every set $M_n(x)$ consists of the edges that can be edges in an n-th derivative of x. Hence,

$$\| f^n(\{ x \}) \subseteq M_n(x),$$

and together with the previous inclusion,

$$\| f^n(\{ x \}) \subseteq M_2(x).$$

In general, one has, for a word $\mathsf{w} \in \mathcal{A}^l$ consisting of letters only that occur in a second derivative of ω, and for a derivation step $n \in \mathbb{N}_0$

$$\begin{aligned}\|f^n(\{\,\mathsf{w}\,\}) =& \|f^n(\{\,w_1\,\}) \cup \|^{\overrightarrow{w_1}(\mathsf{o})} f^n(\{\,w_2\,\}) \cup \cdots \cup \|^{\overrightarrow{w_{l-1}}(\mathsf{o})} f^n(\{\,w_l\,\})\\ &\subseteq M_2(w_1) \cup M_2^{\overrightarrow{w_1}(\mathsf{o})}(w_2) \cup \cdots \cup M_2^{\overrightarrow{w_{l-1}}(\mathsf{o})}(w_l)\\ &= M_2(\mathsf{w}).\end{aligned}$$

Since the set $M_2(\mathsf{w})$ is finite for every word w, there are only finitely many subsets. For any word w, only finitely many sets $\|f^n(\{\,\mathsf{w}\,\})$ with $n \in \mathbb{N}_0$ are different:

$$\forall \mathsf{w} :\ |\{\,\|f^n(\{\,\mathsf{w}\,\}) \mid n \in \mathbb{N}_0\,\}| < \infty.$$

In the sequel, three sets of edge sets are considered. They are called edge systems and are denoted by K_f, K_γ and K_g. Let K_f be defined as the set of those edge sets that belong to derivatives of a derivation step obtained by length constant endomorphisms only:

$$K_f = \{\,\|f^n(\{\,\omega\,\}) \mid n \in \mathbb{N}_0\,\}.$$

Let K_γ be defined as the set of those edge sets that belong to derivatives of a derivation step where the endomorphism applied at last is length contracting:

$$K_\gamma = \{\,\|(h^m \circ g)(\{\,\omega\,\}) \mid m \in \mathbb{N}_0\,\}.$$

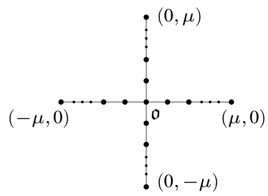
The edge sets that belong to derivatives of a derivation step, where at least one endomorphism applied is a length contracting one, are gathered in the edge system K_g:

$$K_g = \{\,\|(h^m \circ g \circ f^n)(\{\,\omega\,\}) \mid n, m \in \mathbb{N}_0\,\}.$$

The edge system K_f is a set of subsets of $M_2(\omega)$. Since $M_2(\omega)$ is finite, also K_f is finite. According to Lemma 3.6, every edge set $\|(h^m \circ g)(\{\,\omega\,\})$ of K_γ can be written as

$$\|(h^m \circ g)(\{\,\omega\,\}) = \bigcup_{i=1}^{k} \left(\bigcup_{x \in [h^m(\{\,\omega\,\})]} \|g_i(\{\,x\,\}) \right) = \bigcup_{x \in [h^2(\{\,\omega\,\})]} \|g(\{\,x\,\}).$$

Hence, all edges lie on the following cross:



Since, it has only finitely many ‘subcrosses’, the edge system K_γ is finite.
Every edge set $\|(h^m \circ g \circ f^n)(\{\,\omega\,\})$ of K_g satisfies

$$\|(h^m \circ g \circ f^n)(\{\,w\,\}) = \bigcup_{\|\mathsf{v} \in K_\gamma} \|f^n(\{\,\mathsf{v}\,\}) \subseteq \bigcup_{\|\mathsf{v} \in K_\gamma} M_2(\mathsf{v}).$$

Since $M_2(\mathsf{v})$ is finite for any word v with an edge set belonging to K_γ and K_γ itself is finite, the union

$$U = \bigcup_{\|\mathsf{v} \in K_\gamma} M_2(\mathsf{v})$$

is also finite. Every element of K_g is a subset of U. Since the set U is finite, it has only finitely many subsets; hence, K_g has only finitely many elements.

For every derivation step $n \in \mathbb{N}_0$, one has

$$\| f^n(\{\,\omega\,\}) = \bigcup_{\mathsf{w} \in f^n(\{\,\omega\,\})} \|\mathsf{w}$$

and $f^n(\{\,\omega\,\})$ is finite. So, the finiteness of K_f implies the finiteness of the set

$$K'_f = \{\,\|\mathsf{w} \mid \mathsf{w} \in f^n(\{\,\omega\,\}), n \in \mathbb{N}_0\,\}.$$

Similarly, for $m, n \in \mathbb{N}_0$,

$$\|(h^m \circ g \circ f^n)(\{\,\omega\,\}) = \bigcup_{\mathsf{w} \in (h^m \circ g \circ f^n)(\{\,\omega\,\})} \|\mathsf{w}$$

and $(h^m \circ g \circ f^n)(\{\,\omega\,\})$ is finite. The finiteness of K_g implies the finiteness of the set

$$K'_g = \{\,\|\mathsf{w} \mid \mathsf{w} \in (h^m \circ g \circ f^n)(\{\,\omega\,\}), m, n \in \mathbb{N}_0\,\}.$$

The set K_G of all edge sets of words generated by G is the union of K'_f and K'_g:

$$\begin{aligned} K_G &= \{\,\|\mathsf{w} \mid \mathsf{w} \in h^n(\{\,\omega\,\}), n \in \mathbb{N}_0\,\} \\ &= K'_f \cup K'_g. \end{aligned}$$

Thus, K_G is finite. Hence, also the picture language P_G is finite. So, the following lemma is proved.

3.8. Lemma: *If, for every letter $x \in [h^2(\{\,\omega\,\})]$ occurring in a second derivative of ω, the derivatives of x of the first three derivation steps by the length constant endmorphisms of f produce no other x-edge than (o, x), the picture language P_G generated by G is finite:*

$$\forall x \in [h^2(\{\,\omega\,\})]:\ \|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\}) \Longrightarrow |P_G| < \infty.$$

This lemma is the second part of the equivalence supposed above. Thus, together with Lemma 3.5, the supposition is confirmed. The following theorem summarizes this result.

3.9. Theorem: *The picture language P_G generated by G is finite if and only if, for every letter x occurring in a second derivative of ω, the derivatives of x of the first three derivation steps by the length constant endmorphisms of f do not produce an x-edge different from (o, x):*

$$|P_G| < \infty \Longleftrightarrow \forall x \in [h^2(\{\,\omega\,\})]:\ \|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\}).$$

With this condition, one can decide whether a length constant *sT0L* system generates a finite picture language or not.

3.10. Theorem: *It is decidable in linear time whether an sT0L system generates a finite picture language or not.*

Proof: The Theorems 3.3, 3.4 and 3.9 yield the decidability.

The decision is even possible in linear time. This can be seen as follows: For length contracting and length expanding systems, the finiteness is decided in constant time. For length constant systems, the condition given in Theorem 3.9 has to be examined.

For any letter $x \in [h^2(\{\,\omega\,\})]$ (four at most), it is examined whether a derivative x' of x by the length constant endomorphisms produces an x-edge not starting at $\mathfrak{o}$.
If so, then $\|_x x \neq \|_x f(\{\,x\,\})$.

If the equality $\|_x x = \|_x f(\{\,x\,\})$ holds then it is examined whether there is a letter $x \in [h^2(\{\,\omega\,\})]$ for which another letter y exists with $y \in [f(\{\,x\,\})]$ and $x \in [f(\{\,y\,\})]$ such that x yields different y-edges or y yields different x-edges or all derived y-edges start at the same point and the resulting x-edge does not start at $\mathfrak{o}$.
If so, then $\|_x x \neq \|_x f^2(\{\,x\,\})$.

If also the second equality $\|_x x = \|_x f^2(\{\,x\,\})$ holds then it is examined whether a letter $x \in [h^2(\{\,\omega\,\})]$ exists, for which two other letters y and z exist with $y \in [f(\{\,x\,\})]$, $z \in [f(\{\,y\,\})]$ and $x \in [f(\{\,z\,\})]$ such that x yields different y-edges or y yields different z-edges or z yields different x-edges or all derived y-edges start at the same point and the resulting z-edge yields an x-edge not starting at $\mathfrak{o}$.
If so, then $\|_x x \neq \|_x f^3(\{\,x\,\})$.

In all other cases, the equalities $\|_x x = \|_x f(\{\,x\,\}) = \|_x f^2(\{\,x\,\}) = \|_x f^3(\{\,x\,\})$ hold.

Hence, for each occurring derivative, the edges produced are examined constantly often. Thus, the time needed for decision is linear in the size of the system. ♣

4. Conclusion

In the present paper, synchronous, tabled, context-free chain code picture systems based on Lindenmayer systems (*sT0L* systems) are studied with respect to the finiteness of the picture languages generated. It is shown that it is decidable in linear time whether an *sT0L* system generates a finite picture language or not.

The systems considered are divided up in length contracting, constant and expanding systems. Length contracting systems generate finite picture languages. Length expanding systems generate infinite picture languages. Among the length constant systems are those with finite picture languages and those with infinite ones. The paper gives a condition that allows to decide after the third derivation step whether a *sT0L* system generates a finite picture language or an infinite one.

References

[DH89] DASSOW, J.; HINZ, F.: *Kettencode-Bildsprachen. Theorie und Anwendungen*. Wiss. Zeitschrift der Techn. Univ. Magdeburg, 1989.

[DHr92] DASSOW, J.; HROMKOVIČ, J.: *On Synchronized Lindenmayer Picture Languages*. Lindenmayer Systems, 253–261. Springer-Verlag, Berlin 1992.

[Fe68] FEDER, J.: *Languages of encoded line patterns*. Inform. Control, 13:230–244.

[Fr61] FREEMAN, H.: *On the encoding of arbitrary geometric configurations*. IRE Trans. EC, 10:260–168, 1961.

[MRW82] MAURER, H.; ROZENBERG, G.; WELZL, E.: *Using string languages to describe picture languages*. Inform. Control, 54:155–185, 1982.

[RS80] ROZENBERG, G.; SALOMAA, A.: *The Mathematical Theory of L Systems*. Academic Press, 1980.

[SW85] SUDBOROUGH, I. H.; WELZL, E.: *Complexity and decidability for chain code picture languages*. Theoretical Computer Science, 36:173–202, 1985.

[T02] TRUTHE, B.: *On the Finiteness of Picture Languages of Synchronous Deterministic Chain Code Picture Systems*. To appear. (see also *Zur Endlichkeit von Bildsprachen synchroner deterministischer Ketten-Code-Bild-Systeme*. Otto-von-Guericke-Universität Magdeburg, Preprint Nr. 8/2002.)

[T03] TRUTHE, B.: *On the Finiteness of Picture Languages of synchronous, simple nondeterministic Chain Code Picture Systems*. Fundamenta Informaticae, 56:389–409, 2003.

[T04] TRUTHE, B.: *A Method for Deciding the Finiteness of Deterministic Tabled Picture Languages*. In *Proceedings of the Eighth International Conference on Developments in Language Theory*, Springer-Verlag *LNCS* 3340:406–417, 2004.