Higher Order Logic with Henkin Semantics

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0.1 Introduction

This paper consists of a systematical study of the model-theoretic properties of higher order logic with Henkin semantics, in a simplified version (who will be denoted in the following *HNK*). This logic is organized as an institution [14] and we develop a framework for applying known institution-independent model theory results, including borrowing properties along institution mappings, to the particular case of *HNK*.

Higher order logic or type theory was invented by Bertrand Russell [27] in order to provide foundations for mathematics. In [31] various fundamental parts of mathematics are formalized in this system. The term 'higher order' means that we allow variables to range over functions and predicates. In general, the formulation of Church is now used [5]. This variant was proved complete by Henkin [19] and in the same paper the so-called higher order logic with Henkin semantics is introduced. The difference is that types are interpreted not as entire function spaces but as function subspaces instead. We refer to [1] for a complete overview on higher order logics.

From a computer-science perspective, higher order logics may be used for directly proving theorems, when they provide a suitable specification language for the task (e.g. hardware verification, see for example [4] for a motivation, or for describing data structures, see [25]) or as theorem proving support. Widely-used higher-order theorem provers are the HOL system [16] and Isabelle [26]. Since in practice it can be convenient to use only term-denotable subsets of function spaces as carriers for higher order types, higher order logic with Henkin semantics seems an appropriate choice.

The simplifications mentioned before on the higher order logic with Henkin semantics are that we do not allow λ-abstraction as terms, which does not represent a loss in power of expressiveness (because we can add replace them with a new constant symbol and an axiom), and we do not introduce logical connectives and quantifiers in the signature, but keep them in the meta-language. One of the consequences is that the valuation of formulas is possible in any model.

The paper is organized as follows:

After a short preliminary recall of the category theory results used in the paper, the institution of higher order logic with Henkin semantics is introduced and we prove its satisfaction condition. In the next two sections we define two institution comorphisms. The first maps higher order logic with Henkin semantics to the institution of presentations over first order equational logic and will be widely used in the paper for obtaining results about *HNK*. The definition of this comorphism is based on a transcription in the institutional framework of the ideas from [24] and its significance is that the power of expressivity of *HNK* can be obtained in the framework of first order logics only with a certain cost. The second one, mapping first order equational logic to *HNK*, shows the contrary: *HNK* is expressive enough to encode first order logic without any supplementary requirements. The last section contains the analysis of the model-theoretic properties of *HNK*. In each case, we either give a proof or provide a counterexample. The results obtained for *HNK* are extremely useful both from a model-theoretic and computer-science perspective: signature pushouts, weak model amalgamation, quasi-representability for signature extensions with constants, substitutions between signature inclusions, direct products and ultraproducts of models, elementary diagrams and even though we obtain that not all HNK atoms are basic sentences, we show that HNK is a Łoś institution and thus is compact. We also obtain interpolation results for HNK both via axiomatizability and by borrowing along institution comorphisms. To facilitate the reading, we preferred to introduce each concept of institution-independent model theory at the beginning of each paragraph instead of making a separate section.

Categories

We will assume that the reader is familiar with the basic notions of category theory, like functor, natural transformation, colimit etc. We refer to [20] as the standard textbook on this topic and we will follow its terminology, except that we denote the composition ; and write it in a diagrammatic order.

Let $\mathbb{C}$ and $\mathbb{S}$ be two categories such that $\mathbb{S}$ is small. A functor $D : \mathbb{S} \longrightarrow \mathbb{C}$ is also called a *diagram*. We usually identify a diagram $D : \mathbb{S} \longrightarrow \mathbb{C}$ with its image in $\mathbb{C}$, $D(\mathbb{S})$. Any set with a partial order defined on it $(J, \leq)$ can be regarded as a category in the obvious way, with the arrows being pairs $i \leq j$. $(J, \leq)$ is said to be *directed* if for all $i, j \in J$, there exists $k \in J$ such that $i \leq k$ and $j \leq k$. A diagram defined on a directed set will be called *directed diagram* , and a colimit of such a diagram *directed colimit* .

Given a functor $U : \mathbb{C}' \to \mathbb{C}$, for any object $A \in |\mathbb{C}|$ we define *the comma category* A/U which has as objects the arrows from $\mathbb{C}$ of form $f : A \to U(B)$, sometimes denoted as (f, B) and $A/U((f, B), (f', B')) = \{h \in \mathbb{C}'(B, B') | f; U(h) = f'\}$. If $\mathbb{C} = \mathbb{C}'$ and $U = 1_{\mathbb{C}}$, we denote $A/U = A/\mathbb{C}$.

An *indexed category* is a functor $B : I^{op} \to \mathbb{C}at$. Sometimes, $B(i)$ is denoted $\mathbb{B}^i$ for $i \in |I|$ and similarly $B(u)$ is denoted B^u for $i \in I(i, j)$. We define *the Grothendieck category* over B, denoted $\mathbb{B}^\sharp$, as the category that has as objects pairs $< i, \Sigma >$ with $i \in |I|$ and $\Sigma \in |\mathbb{B}^i|$ and arrows from $< i, \Sigma >$ to $< i', \Sigma' >$ pairs $< u, \varphi >$ with $i \in I(i, i')$ and $u \in \mathbb{B}^i(\Sigma, B^u(\Sigma'))$.

Theorem 0.1 *[30]*

Given an indexed category $B : I^{op} \to \mathbb{C}at$, then for each category J the Grothendieck category $\mathbb{B}^\sharp$ has:

- *J-limits when I has J-limits, $\mathbb{B}^i$ has J-limits for each index i and B^u preserves J-limits for each index morphism u and*
- *J-colimits when I has J-limits, $\mathbb{B}^i$ has J-limits for each index i and B^u has a left adjoint for each index morphism u.*

Definition 1 *An adjunction from the category $\mathbb{X}$ to the category $\mathbb{A}$ consists of a tuple $(U, F, \eta, \varepsilon)$ such that $U : \mathbb{A} \to \mathbb{X}$ and $F : \mathbb{X} \to \mathbb{A}$ are functors and $eta : 1_{\mathbb{X}} \to F; U$ and $\varepsilon : U; F \to 1_{\mathbb{A}}$ are natural transformation such that the following equations hold: $\eta F; F\eta = 1_F$ and $U\eta; \varepsilon U = 1_U$.*

Given an adjunction $(U, F, \eta, \varepsilon)$, for any object $X \in |\mathbb{X}|$ there exists an object XF called U-free over $\mathbb{A}$ and an arrow $\eta_X : X \to U(F(X))$ such that for each object $A \in |\mathbb{A}|$ and any arrow $h : X \to U(A)$ there exists an unique arrow $h' : F(X) \to A$ such that $h = \eta_X; U(h')$.

The object $F(X)$ is persistently U-free when η_X is an isomorphism and the adjunction is persistent if for each object X of $\mathbb{X}$ $F(X)$ is persistently U-free.

0.2 Institutions

Institutions were introduced by Goguen and Burstall [14] to formally capture the informal notion of logical system from a model-oriented perspective. The original goal was to provide an abstract, logic-independent framework for algebraic specifications of computer science systems, since any such algebraic specifications formalism relies on a logical system allowing the user to describe the properties of the software to be developed. The software itself is represented as a model of the system, and a notion of satisfaction determines whether an axiom holds in a model, and therefore for the system the model represents. It is natural to develop a theory of specification formalisms in a way that is as much as possible independent from the choice of underlying system: this would not only bring a separation of different issues (details of a particular logic and general concepts, logic independent) but it would also allow to apply the abstract results of the theory to a certain formalism well suited for a given task.

Since they were defined, institutions gained the position of major tool in the development of the theory of specification and it became standard in the field to express the logical system underlying a particular language or system in the language of the theory of institutions (see CASL[2] or CafeOBJ[12]).

Besides its importance for algebraic specifications, the theory of institutions also provides an appropriate level of generality for the development of abstract model theory, by offering an uniform approach to the model theory of various logics and facilitating a deeper understanding of model theoretic phenomena. One can obtain new non-trivial results for non-classsical logics in a considerably easier manner. Institution independent model theory also provides an efficient framework for translating properties along mappings between institutions.

We cite the following papers among the most relevant results on abstract model theory: [11], [13], [23], [18], [17], [9], [8], [7], [22].

Definition 2 *An institution $I = (\mathbb{S}ign, Sen, Mod, \models)$ [14] consists of:*

- *a category $\mathbb{S}ign$, whose objects are called signatures and whose arrows are called signature morphisms;*
- *a functor $Sen : \mathbb{S}ign \to \mathbb{S}et$, (corresponding intuitively to the syntax of the logic) that assigns to each signature a set called the set of sentences over that signature;*
- *a functor $Mod : \mathbb{S}ign^{op} \to \mathbb{C}at$, (giving the semantics of the logic) such that for any signature Σ, the objects of $Mod(\Sigma)$ are called Σ-models or just models and the arrows of $Mod(\Sigma)$ are called model homomorphisms;*
- *a binary relation $\models = \{\models_\Sigma : |Mod(\Sigma)| \times Sen(\Sigma) | \Sigma \in Sign\}$ called the satisfaction relation*

such that the following satisfaction condition holds:

$$Mod(\varphi)(M') \models_\Sigma e \iff M' \models_{\Sigma'} Sen(\varphi)(e)$$

for each signature morphism $\varphi : \Sigma \to \Sigma'$, each Σ-sentence e and each Σ'-model M'.

Notice that this formalization only assumes abstract categories or classes of signatures, sentences and models, without any constraint on their structure. The only requirement is the satisfaction condition, with the meaning that truth is invariant under change of notation and enlargement of context.

For a signature morphism $\varphi : \Sigma \to \Sigma'$, we sometimes denote $Sen(\varphi)(e) = \varphi(e)$ for any Σ-sentence e and $Mod(\varphi)(M) = M \upharpoonright_\varphi$ for any Σ'-model M, where $Mod(\varphi)$ is the functor determined by φ between $Mod(\Sigma')$ and $Mod(\Sigma)$, called the model reduct functor. For any Σ'-model M' and any Σ-model M, when $M = M' \upharpoonright_\varphi$ we say that M is the reduct of M' and M' is an expansion of M.

Examples of institutions.

1. First order logic with equality(denoted FOL).

   We quote [28] for an introduction to classical first order logic. FOL was first organized as an institution in [14].

   Signatures. Signatures are triples (S,F,P), where S is a set (of sorts), $F = \cup_{w \in S^*, s \in S} F_{w \to s}$ is the collection of operation symbols gruped by their arity $w \in S^*$ and their rank $s \in s$ and $P = \cup_{w \in S^*} P_w$ is the collection of predicate symbols, also grouped by arity.

   A signature morphism $\varphi : (S,F,P) \to (S',F',P')$ consists of a mapping φ between sorts and for each $w \in S^*,\ s \in S$ a mapping between $F_{w \to s}$ and $F'_{\varphi^*(w) \to \varphi(s)}$ and a mapping between P_w and $P'_{\varphi^*(w)}$, where if $w = s_1..s_n$, $\varphi^*(w) = \varphi(s_1)...\varphi(s_n)$.

   Models. A model M of a signature (S,F,P) interpret sorts as sets, operation symbols as functions such that if $\sigma \in F_{w \to s}$, $M_\sigma : M_w \to M_s$ and predicate symbols $\pi \in P_w$ as subsets $M_\pi \subseteq M_w$, where if $w = s_1..s_n$, $M_w = M_{s_1} \times ... \times M_{s_n}$.

   A model homomorphism $h : M \to N$ is an S-sorted function $\{h : M_s \to N_s | s \in S\}$ that preserves both operation and predicate symbols: $h_s(M_\sigma(m_1,..,m_n)) = N_\sigma(h_{s_1}(m_1),...,h_{s_n}(m_n))$ for any operation symbol $\sigma \in F_{s_1...s_n \to s}$ and any $m_i \in M_{s_i}$ and $h_w(M_\pi) \subseteq N_\pi$ for any predicate symbol $\pi \in P_w$.

   For each signature morphism $\varphi : \Sigma \to \Sigma'$, the functor $Mod(\varphi) = (\_ \upharpoonright_\varphi)$ assigns to each Σ'-model M' a Σ-model M such that $M_x = M'_{\varphi(x)}$, where x stands for each sort, operation symbol or predicate symbol, and to each Σ'-model homomorphism $h' : M' \to N'$ the model homomorphism $h' \upharpoonright_\varphi : M' \upharpoonright_\varphi \to N' \upharpoonright_\varphi$ defined by $(h' \upharpoonright_\varphi)_s = h'_{\varphi(s)}$.

   Sentences. Given a signature (S,F,P), we define the F-terms inductively: each $\sigma \in F_{\to s}$ is a term of sort s and for each $\sigma \in F_{w \to s}$, $\sigma(t_1,..,t_n)$ is a term of sort s if t_i are terms of sort s_i. The atomic formulae are either of form $t = t'$, where t,t' are terms of the same sort or $\pi(t_1,..,t_n)$, where t_i is a term of sort s_i. The set of (S,F,P)-sentences is the least set that contains the atoms and is closed under Boolean connectives and quantification. A quantified sentence by a finite set of variables X is of form $(\forall X)\rho$ where ρ is a $(S,F \uplus X,P)$-sentence and we added to the signature the variables as new constants.

   The sentence translation along a signature morphism $\varphi : \Sigma \to \Sigma'$ is defined inductively on the structure of sentences by replacing the symbols from Σ with their corresponding symbols by φ in Σ'. The only thing that requires attention is that when translating a variable symbol of sort s, it becomes a variable symbol of sort $\varphi(s)$.

Satisfaction. Each term $t = \sigma(t_1,..t_n)$ is interpreted in the model M as $M_\sigma(M_{t_1},..,M_{t_n})$.

The satisfaction relation between models and sentences is defined inductively on the structure of sentences. For a fixed signature (S,F,P):

- $M \models t = t'$ if $M_t = M'_t$;
- $M \models \pi(t_1,..,t_n)$ if $(M_{t_1},..M_{t_n}) \in M_\pi$;
- $M \models e_1 \wedge e_2$ if and only if $M \models e_1$ and $M \models e_2$, and similarly for all Boolean connectives,
- $M \models (\forall X)e$ if for each $(S,F \uplus X,P)$-expansion M', $M' \models e$ and similarly for existential quantification.

One can show that the satifaction condition holds, which defines completely the institution of the first order logic with equality.

2. The institution of first order equational logic with equality, denoted $FOEQL$, is obtained from FOL by discarding the predicate symbols from all signatures and their interpretations in models.

3. *The institution of presentations I^p over a base institution I.*

   Given an institution $I = (\mathbb{S}ign, Sen, Mod, \models)$, a presentation is a pair (Σ,E), with $\Sigma \in |Sign|$ and $E \subseteq Sen(\Sigma)$. A presentation morphism $\varphi : (\Sigma,E) \to (\Sigma',E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \varphi(E)$(where we denote by $\models$ the relation of semantical consequence between sets of sentences - for any two sets of sentences $E,E' \in Sen(\Sigma)$, $E \models E'$ if and only if any Σ-model M of E is also a model of E').

   We define the institution of the presentations I^p over the base institution as follows:

   - $\mathbb{S}ign^p$ is the category $\mathbb{P}res$ of the presentations of I;
   - for each presentation (Σ,E), $Mod^p(\Sigma,E) = Mod(\Sigma,E)$;
   - for each presentation (Σ,E), $Sen^p(\Sigma,E) = Sen(\Sigma)$;
   - $M \models^p_{(\Sigma,E)} e \iff M \models_\Sigma e$, for each (Σ,E)-model M and each (Σ,E)-sentence e.

0.3 The institution of higher order logic with Henkin semantics

Higher order logic with Henkin semantics was introduced in [19], where also a completeness result is proven. What is particular to this logic is that, unlike the original version of higher order logic(see [5] for a formulation), where a type $s \to s'$ is interpreted in a model D as the entire function space $[D_s \to D_{s'}]$, types are interpreted as function subspaces of $[D_s \to D_{s'}]$.

The institution we present here follows the conventions from [24]; as same as there, we do not allow λ-abstractions as terms, since any λ-abstraction $\lambda x.t$ may be replaced with a new constant symbol f together with the axiom $fx = t$. Therefore, using the terminology of [19], the models of our institution are not frames, but general models, since any formula may be evaluated in a model (this is true also because the logical connectors are not introduced as symbols in the signature, but are contained in the meta-language).

We quote as related work the paper of Borzyszkowski [3], where the higher order logic used in theorem provers like Isabelle [26] and the HOL system is organized as an institution.

0.3.1 Signatures

Definition 3 *A signature in higher order logic with Henkin semantics is a pair (S,F) where:*

- *S is a set (of sorts). We define the set of types over S, denoted $\overrightarrow{S}$, as the least set that contains S and for any $s,s' \in \overrightarrow{S}$, $s \to s' \in \overrightarrow{S}$.*
- *$F = \bigcup_{s\in\overrightarrow{S}} F_s$, where F_s is a set (of constant symbols of type s), for every $s \in \overrightarrow{S}$.*

Definition 4 *Given two signatures, (S,F) and (S',F'), a signature morphism $\varphi : (S,F) \to (S',F')$ consists of:*

- *a sort function $\varphi^{sort} : S \to S'$ and*
- *a function between operation symbols $\varphi^{op} : F \to F'$ such that $\varphi^{op}(F_s) \subseteq F'_{\varphi^{type}(s)}$,*

where $\varphi^{type} : \overrightarrow{S} \to \overrightarrow{S'}$ is the extension of φ^{sort} to types inductively defined by $\varphi^{type}(s) = \varphi^{sort}(s)$, for any $s \in S$ and $\varphi^{type}(s \to s') = \varphi^{type}(s) \to \varphi^{type}(s')$, for any $s,s' \in \overrightarrow{S}$.

We may omit superscripts to facilitate reading.

If the sort function of a signature morphism $\varphi : (S,F) \to (S',F')$ assigns to each sort of S a type of $\overrightarrow{S'}$ instead of a type over S' (i.e. $\varphi^{sort} : S \to \overrightarrow{S'}$), the signature morphism will be called *type-derived*.

Fact 0.2 *The signatures and the signature morphisms form a category, denoted $\mathbb{S}ign^{HNK}$, under the obvious composition of signature morphisms.*

Remark 1 *The type-derived signature morphisms also form a category.*

0.3.2 Models

Definition 5 *Given a signature (S,F), a model M interprets:*

- *each sort $s \in S$ as a set, M_s;*
- *each type $s \to s' \in \overrightarrow{S}$ as a subset $M_{s\to s'} \subseteq [M_s \to M_{s'}]$, where $[M_s \to M_{s'}] = \{f : M_s \to M_{s'} | f\ function\}$;*
- *each constant symbol $\sigma \in F_s$, where $s \in \overrightarrow{S}$, as an element of M_s.*

<u>Convention:</u> The interpretation of a type s in a model M is assumed empty if it is not defined explicitly.

Definition 6 *A (S,F)-model homomorphism $h: M \to N$ is a $\overrightarrow{S}$-sorted function $\{h_s : M_s \to N_s\}_{s \in \overrightarrow{S}}$ such that $h_s(M_\sigma) = N_\sigma$ for any $s \in \overrightarrow{S}$ and any $\sigma \in F_s$ and $h_{s'}(f(x)) = h_{s\to s'}(f)(h_s(x))$, for any $s \to s' \in \overrightarrow{S}$, any $x \in M_s$ and any $f \in M_{s\to s'}$.*

$$\begin{array}{ccc} M_s & \xrightarrow{f} & M_{s'} \\ {\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h_{s'}} \\ N_s & \xrightarrow[h_{s\to s'}(f)]{} & N_{s'} \end{array}$$

Fact 0.3 *For any signature (S,F), the (S,F)-models and (S,F)-model homomorphisms form a category, $Mod^{HNK}(S,F)$, where the composition of homomorphisms is made component-wise.*

For any signature morphism $\varphi : (S,F) \to (S',F')$ and any (S',F')-model M', we define $Mod^{HNK}(\varphi)(M') = M' \upharpoonright_\varphi$ by $(M' \upharpoonright_\varphi)_s = M'_{\varphi^{type}(s)}$, for any type s and $(M' \upharpoonright_\varphi)_\sigma = M'_{\varphi^{op}(\sigma)}$ for any operation symbol $\sigma \in F$. Each model homomorphism $h : M \to N$ is mapped to $Mod(\varphi)(h) = h \upharpoonright_\varphi : M \upharpoonright_\varphi \to N \upharpoonright_\varphi$, defined by $(h \upharpoonright_\varphi)_s(x) = h_{\varphi^{type}(s)}(x)$ for each type s.

Fact 0.4 *For each signature morphism $\varphi : (S,F) \to (S',F')$, $Mod^{HNK}(\varphi) : Mod^{HNK}(S',F') \to Mod^{HNK}(S,F)$ is a functor.*

Moreover, $Mod^{HNK} : \mathbb{S}ign^{op} \to \mathbb{C}at$ is a functor.

0.3.3 Sentences and satisfaction

Definition 7 *Let (S,F) be a HNK signature. For each $s \in \overrightarrow{S}$, an operation symbol $\sigma \in F_{\to s}$ is a term of type s and $t(t')$ is a term of type s_1 if t is a term of type $s \to s_1$ and t' is a term of type s.*

We denote T_F the $\overrightarrow{S}$-sorted set of F-terms.

The interpretation of terms in a model is defined inductively, by extending the interpretation of operation symbols: if $t(t')$ is a term of type s' and M is a model, $M_{t(t')} = M_t(M_{t'})$.

The atomic sentences are equations of form $t = t'$, where t, t' are terms of the same type. The HNK-sentences are obtained from the atomic sentences by using the usual Boolean connectives and higher order quantification.

The sentence translation $Sen^{HNK}(\varphi) : Sen^{HNK}(S,F) \to Sen^{HNK}(S',F')$ along a signature morphism $\varphi : (S,F) \to (S',F')$ is defined by using the function $\varphi : T_F \to T'_F$ induced by φ. Then

- $Sen^{HNK}(\varphi)(t = t') = (\varphi(t) = \varphi(t'))$;
- $Sen^{HNK}(\varphi)(\neg e) = \neg Sen^{HNK}(\varphi)(e)$, and similarly for all Boolean connectives;
- $Sen^{HNK}(\varphi)((\forall X)e) = (\forall X^\varphi)Sen^{HNK}(\varphi')(e)$, where X is a finite $\overrightarrow{S}$-sorted set of variables(because we admit quantification over variables of any higher-order type), $X^\varphi_s = \bigcup_{\varphi(s')=s} X_{s'}$ and $\varphi' : (S, F \uplus X) \to (S', F' \uplus X^\varphi)$ extends φ canonically.

We denote $Sen^{HNK}(\varphi)(e) = \varphi(e)$.

Fact 0.5 *$Sen : Sign \to Set$ is a functor.*

The satisfaction relation is defined inductively on the structure of sentence. Given a signature (S,F):

- $M \models t = t'$ if and only if $M_t = M_{t'}$;
- $M \models e_1 \wedge e_2$ if and only if $M \models e_1$ and $M \models e_2$, and similar for all Boolean connectives;
- $M \models (\forall X)e'$ if and only if for each expansion M' of M along the signature inclusion $(S,F) \hookrightarrow (S, F \uplus X)$, $M' \models e'$.

Proposition 0.6 *For any signature morphism $\varphi : (S,F) \to (S',F')$, any $M' \in |Mod^{HNK}(S,F)|$ and any $e \in Sen^{HNK}(S,F)$, $M' \restriction_\varphi \models e$ if and only if $M' \models \varphi(e)$.*

Proof: We begin with a lemma which we prove later:

Lemma 0.7 *For each signature morphism $\varphi : (S,F) \to (S',F')$, each (S',F')-model M' and each term $t \in T_F$, $(M' \restriction_\varphi)_t = M'_{\varphi(t)}$.*

The proof of the proposition will be given by induction on the structure of e.

For the basic case, assume that $e = (t = t')$. Then $M' \models \varphi(t = t') \iff M' \models \varphi(t) = \varphi(t') \iff M'_{\varphi(t)} = M'_{\varphi(t')} \iff$ (by using the lemma) $(M' \restriction_\varphi)_t = (M' \restriction_\varphi)_{t'} \iff M' \restriction_\varphi \models t = t'$.

For the general case, we consider only the nontrivial subcase of universal quantification. Suppose that $e = (\forall X)e'$. We show that $M' \restriction_\varphi \models (\forall X)e' \iff M' \models \forall(X^\varphi)\varphi'(e')$:

$$\begin{array}{ccc}
(S,F) & \xrightarrow{\varphi} & (S',F') \\
\big\downarrow & & \big\downarrow \\
(S, F \cup X) & \xrightarrow{\varphi'} & (S', F' \cup X^\varphi)
\end{array}$$

For the direct implication, let N' be a X^φ-expansion of M', $N = N' \restriction_{\varphi'}$ and $M = M' \restriction_\varphi$. Because $N' \restriction_{X^\varphi} \restriction_\varphi = M$ and the diagram commutes, we have that $N \restriction_X = M$, so by using the hypothesis $N \models e'$, which implies by using the induction hypothesis $N' \models \varphi'(e')$.

For the converse, let N be a X-expansion of M. We define an expansion for M', denoted N', by interpreting $x \in X^\varphi$ as N_x. We check that the definition is correct: because $N_x \in N_s$, we also have that $N_x \in M_s$, so by using the definition of the reduct, $N_x \in M'_{\varphi(s)} = N'_{\varphi(s)}$. By using the hypothesis we have that $N' \models \varphi'(e')$, so $N \models e'$ by the induction hypothesis.

Proof of lemma

Induction on the structure of term.

If $t = \sigma$ with $\sigma \in F$, by definition of the reduct functor we have that $(M' \restriction_\varphi)_\sigma = M'_{\varphi(\sigma)}$.

Assume that $t = t_1(t_2)$ and the conclusion holds for t_1 and t_2. Then $(M' \restriction_\varphi)_{t_1(t_2)} =$ $(M' \restriction_\varphi)_{t_1}((M' \restriction_\varphi)_{t_2}) =$(by using the induction hypothesis)$M'_{\varphi(t_1)}(M'_{\varphi(t_2)}) = M'_{\varphi(t_1)(\varphi(t_2))} = M'_{\varphi(t_1(t_2))}$.

■

0.3.4 The institution of higher order logic

The institution of higher-order logic, denoted HOL, is obtained from HNK by restricting the models to those that interpret each type $s \to s'$ as the entire function space $[M_s \to M_{s'}]$ instead of a subset of it.

We check that HOL is an institution, which means the functor Mod maps HOL-models into HOL-models. Let $\varphi : (S,F) \to (S',F')$ be a signature morphism and let M' be a HOL (S',F')-model. We denote $M = M' \upharpoonright_{\varphi}$ and prove that $M_{s\to s'} = [M_s \to M_{s'}]$ for any types s, s'. $M_{s\to s'} = M'_{\varphi(s)\to\varphi(s')} = [M'_{\varphi(s)} \to M'_{\varphi(s')}] = [M_s \to M'_{s'}]$.

0.4 Comorphism $HNK \to FOEQL^p$

Institution comorphisms

The original paper about institutions [14] introduces the notion of institution morphism, concept which includes structure forgetting and arises naturally. Because not all important relationships between institutions are captured, the dual concept of comorphism [15], introduced first in [21] under the name of 'plain map', then under the name of 'representation' [29], is also meaningful. Institution comorphisms formalize an embedding or a encoding of the source institution to the target one.

Definition 8 *Given two institutions, $I = (\mathbb{S}ign, Sen, Mod, \models)$ and $I' = (\mathbb{S}ign', Sen', Mod', \models')$, an institution comorphism [21] [15] $(\phi, \alpha, \beta) : I \to I'$ consists of:*

- *a functor $\phi : \mathbb{S}ign \to \mathbb{S}ign'$;*
- *a natural transformation $\alpha : Sen \Rightarrow \phi; Sen'$;*
- *a natural transformation $\beta : \phi^{op}; Mod' \Rightarrow Mod$;*

such that the following satisfaction condition holds

$$M' \models'_{\phi(\Sigma)} \alpha_\Sigma(e) \iff \beta_\Sigma(M') \models_\Sigma e$$

for each signature $\Sigma \in |\mathbb{S}ign|$, each Σ-sentence e and each $\phi(\Sigma)$-model M'.

As an example, we briefly describe the institution comorphism from FOL to $FOEQL$ that encodes the relation symbols of a FOL signature as operation symbols in the corresponding $FOEQL$ signature. Each signature (S,F,P) is mapped to $(S \uplus \{b\}, F \uplus \overline{P} \uplus \{true\})$, where b is a new sort, $true$ is a constant symbol of sort b, $\overline{P}_{w\to s} = P_w$ if $s = b$ or the empty set otherwise. Each relational atom $\pi(t)$ is mapped to $\pi(t) = true$. For each $(S \uplus \{b\}, F \uplus \overline{P} \uplus \{true\})$-model A, $\beta(A)$ interprets the sorts and the operations like A and $\beta(A)_\pi = A_\pi^{-1}(A_{true})$. One can prove that we obtain thus an institution comorphism.

We define an institution comorphism $(\phi, \alpha, \beta) : HNK \to FOEQL^p$. Its significance is that one can encode the institution of higher order logic with Henkin semantics into first order equational logic, but the only algebras that are the image of HNK-models under the comorphism are those satisfying the axioms of extensionality. Because the latter do not fall into any usual format (for example they are not Horn clauses), an analysis of the properties of the class of models of the presentations $\Gamma_{(S,F)}$ is not trivial. This comorphism is very useful for borrowing results along it, since we will show that $\alpha_{(S,F)}$ is isomorphism and $\beta_{(S,F)}$ is an equivalence of categories, for any HNK signature (S,F).

- The functor $\phi : \mathbb{S}ign^{HNK} \to \mathbb{P}res^{FOEQL}$:

  Each HNK signature (S,F) is mapped to a $FOEQL$ presentation $\phi(S,F) = ((\overrightarrow{S}, \overrightarrow{F}), \Gamma_{(S,F)})$, where $\overrightarrow{S}$ is the set of types over the sorts in S, $\overrightarrow{F}_{\to s} = F_s$ for any type $s \in \overrightarrow{S}$, $\overrightarrow{F}_{(s\to s',s)\to s'} = \{apply_{s\to s'}\}$ and $\overrightarrow{F}_x = \emptyset$ otherwise. The set of axioms for each presentation state that the models are *extensional*, i.e. two elements cannot be differrent if they act the same when 'applied' to the same element:

  $$\Gamma_{(S,F)} = \{\ (\forall f)(\forall g)[((\forall x)apply_{s\to s'}(f,x) = apply_{s\to s'}(g,x)) \implies f = g] | s,s' \in \overrightarrow{S}\ \}$$

  For each signature morphism $\varphi : (S,F) \to (S',F')$, we define $\phi(\varphi) : (\overrightarrow{S}, \overrightarrow{F}) \to (\overrightarrow{S'}, \overrightarrow{F'})$ as follows:

  - $\phi(\varphi)(s) = \varphi^{type}(s)$, for each type $s \in \overrightarrow{S}$(notice that this implies $\phi(\varphi)(s) = \varphi(s)$ on sorts);
  - $\phi(\varphi)(\sigma) = \varphi(\sigma)$, for each constant symbol $\sigma \in F$;
  - $\phi(\varphi)(apply_{s\to s'}) = apply_{\varphi^{type}(s)\to\varphi^{type}(s')}$.

  Notice that $\phi(\varphi)$ is indeed a presentation morphism.

  Fact 0.8 *$\phi : \mathbb{S}ign^{HNK} \to \mathbb{P}res^{FOEQL}$ is a functor.*

  Remark 2 *The functor $\phi : Sign^{HNK} \to Pres^{FOEQL}$ does not preserve pushouts.*

  Proof:

  Let us consider the following pushout of signature inclusions in HNK:

  $$\begin{array}{ccc} (\{s\},\emptyset) & \longrightarrow & (\{s,a\},\emptyset) \\ \downarrow & & \downarrow \\ (\{s,b\},\emptyset) & \longrightarrow & (\{s,a,b\},\emptyset) \end{array}$$

  By applying ϕ we get the following square of signature inclusions in $FOEQL$:

  $$\begin{array}{ccc} (\overrightarrow{\{s\}}, \overrightarrow{\emptyset}) & \longrightarrow & (\overrightarrow{\{s,a\}}, \overrightarrow{\emptyset}) \\ \downarrow & & \downarrow \\ (\overrightarrow{\{s,b\}}, \overrightarrow{\emptyset}) & \longrightarrow & (\overrightarrow{\{s,a,b\}}, \overrightarrow{\emptyset}) \end{array}$$

  which is not a pushout because

$$\begin{array}{ccc} \overrightarrow{\{s\}} & \longrightarrow & \overrightarrow{\{s,a\}} \\ \downarrow & & \downarrow \\ \overrightarrow{\{s,b\}} & \longrightarrow & \overrightarrow{\{s,a,b\}} \end{array}$$

is not a pushout in $\mathbb{S}et$. ■

- The natural transformation α is the canonical extension of the mapping on terms α^{tm} defined by $\alpha^{tm}(t(t')) = apply_{s\to s'}(\alpha^{tm}(t), \alpha^{tm}(t'))$, where t is a term of type $s \to s'$ and t' a term of sort s.

- Given a signature (S,F), we denote $\beta_{(S,F)}(M) = \overline{M}$ for each (S,F)-model M and $\beta_{(S,F)}(h) = \overline{h}$ for each (S,F)-model homomorphism h.

  For each model M, $\overline{M}$ is inductively defined by the isomorphism $fun^M : M \to \overline{M}$.

  - On each sort $s \in S$, $\overline{M}_s = M_s$ and fun^M_s is the identity;
  - On each type $s \to s' \in \overrightarrow{S}$, assuming that fun^M_s and $fun^M_{s'}$ have been defined, we must interpret $\overline{M}_{s\to s'}$ as a subset of $[\overline{M}_s \to \overline{M}_{s'}]$.

    The isomorphisms fun^M_s and $fun^M_{s'}$ determine an isomorphism between $[M_s \to M'_s]$ and $[\overline{M}_s \to \overline{M}_{s'}]$, denoted $fun^M_{s\to s'}$, such that $fun^M_{s\to s'}(f)(fun^M_s(x)) = fun^M_{s'}(f(x))$ for each $f \in [M_s \to M'_s]$ and each $x \in M_s$.

    Notice that to any $g \in M_{s\to s'}$ we can associate a function, also denoted $g : M_s \to M_{s'}$, defined by $g(x) = M_{apply_{s\to s'}}(g,x)$ for any $x \in M_s$. Thus $M_{s\to s'}$ is in a bijective correspondence with a subset of the function space $[M_s \to M_{s'}]$, denoted $M^{[]}_{s\to s'}$.

    We define $\overline{M}_{s\to s'} = fun^M_{s\to s'}(M^{[]}_{s\to s'})$ and we restrict and co-restrict $fun^M_{s\to s'}$ to $M_{s\to s'}$ and $\overline{M}_{s\to s'}$ respectively. Notice that $fun^M_{s\to s'}(f)(fun^M_s(x)) = fun^M_{s'}(M_{apply_{s\to s'}}(f,x))$.
  - Finally, $\overline{M}_\sigma = fun^M_s(M_\sigma)$ for each $\sigma \in F_s$ and any type $s \in \overrightarrow{S}$ - notice that this means $\overline{M}_\sigma = M_\sigma$ if $\sigma \in F_s$ and s is a sort, not a proper type.

Remark 3 *For any signature morphism $\varphi : (S,F) \to (S',F')$ and for any (S',F')-model M', $fun^{M'\restriction_\varphi}_s = fun^{M'}_{\varphi(s)}$, for each type $s \in \overrightarrow{S}$.*

Proof:

Induction on the type s. If s is a sort, then $fun^{M'\restriction_\varphi}_s = 1_{(M'\restriction_\varphi)_s} = 1_{M'_{\varphi(s)}} = fun^{M'}_{\varphi(s)}$.

Let $s \to s'$ be a type such that the inductive hypothesis holds for s and s'. By definition, $fun^{M'\restriction_\varphi}_s$ is a function with the domain $(M' \restriction_\varphi)_{s\to s'}$ and codomain $(\overline{M'} \restriction_\varphi)_{s\to s'}$. By the definition of the reduct, this means that the domain is $M'_{\varphi(s)\to\varphi(s')}$. By using the definition of fun, we have that for any $f \in M'_{\varphi(s)\to\varphi(s')}$ and any $x \in M'_{\varphi(s)}$, $fun^{M'\restriction_\varphi}_{s\to s'}(f)(fun^{M'\restriction_\varphi}_s(x)) = fun^{M'\restriction_\varphi}_{s'}((M' \restriction_\varphi)_{apply_{s\to s'}}(f,x)) = fun^{M'\restriction_\varphi}_{s'}(M'_{apply_{\varphi(s)\to\varphi(s')}}(f,x))$ = (by induction hypothesis for s') $= fun^{M'}_{\varphi(s')}(M'_{apply_{\varphi(s)\to\varphi(s')}}(f,x)) =$

$fun^{M'}_{\varphi(s)\to\varphi(s')}(f)(fun^{M'}_{\varphi(s)}(x))$. By induction hypothesis for s, $fun_s^{M'\upharpoonright_\varphi}(x) = fun^{M'}_{\varphi(s)}(x)$, so the functions $fun^{M'\upharpoonright_\varphi}_{s\to s'}(f)$ and $fun^{M'}_{\varphi(s)\to\varphi(s')}(f)$ are equal. It follows that $fun^{M'\upharpoonright_\varphi}_{s\to s'}$ and $fun^{M'}_{\varphi}(s\to s')$ are equal.

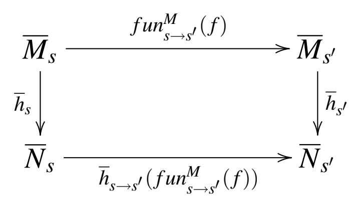
■

For each model homomorphism $h : M \to N$, we define $\overline{h} : \overline{M} \to \overline{N}$ as follows:

- $\overline{h}_s(fun_s^M(x)) = fun_{s'}^N(h(x))$, for any type $s \in \overrightarrow{S}$.

We check that $\overline{h}$ is indeed a model homomorphism. Let $\sigma \in F_s$ be an operation symbol. Then $\overline{h}(\overline{M}_\sigma) = \overline{h}(fun_s^M(M_\sigma)) = fun_s^N(h(M_\sigma))$ =(h is a model homomorphism)= $fun_s^N(N_\sigma) = \overline{N}_\sigma$.

Let $f \in M_s \to M_{s'}$ and $x \in M_s$. We must prove that $\overline{h}_{s\to s'}(fun^M_{s\to s'}(f))(\overline{h}_s(fun_s^M(x))) =$
$\overline{h}_{s'}(fun^M_{s\to s'}(f)(fun_s^M(x)))$.

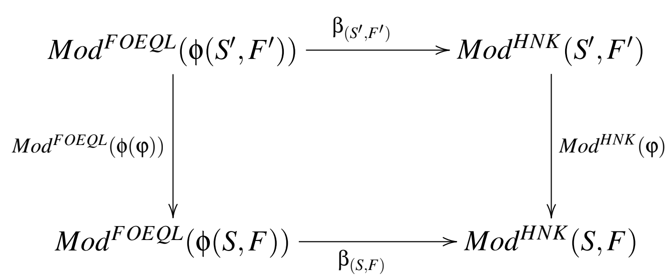


$\overline{h}_{s\to s'}(fun^M_{s\to s'}(f))(\overline{h}_s(fun_s^M(x))) = fun^N_{s\to s'}(h_{s\to s'}(f))(fun_s^N(h_s(x))) = fun^N_{s'}(N_{apply_{s\to s'}}(h_{s\to s'}(f), h_s(x))) =$
$fun^N_{s'}(h_{s'}(M_{apply_{s\to s'}}(f,x))) = \overline{h}_{s'}(fun^M_{s'}(M_{apply_{s\to s'}}(f,x))) = \overline{h}_{s'}(fun^M_{s\to s'}(f)(fun^M_s(x)))$.

Fact 0.9 *For each HNK-signature (S,F), $\beta_{(S,F)} : Mod^{FOEQL}(\phi(S,F)) \to Mod^{HNK}(S,F)$ is a functor.*

Proposition 0.10 $\beta : \phi^{op}; Mod^{FOEQL} \to Mod^{HNK}$ *is a natural transformation.*

Proof:



Let M' be a model in $Mod^{FOEQL}(\phi(S',F'))$. We denote:

- $M = M' \upharpoonright_{\phi(\varphi)}$;
- $\overline{M'} = \beta_{(S',F')}(M')$;

– $\overline{M} = \beta_{(S,F)}(M)$;

We check that $\overline{M} = \overline{M'} \upharpoonright_{\phi}$.

For each sort $s \in S$, $\overline{M}_s = M_s = M'_{\phi(\varphi)(s)} = M'_{\varphi(s)}$. $(\overline{M'} \upharpoonright_{\phi})_s = \overline{M'}_{\varphi(s)} = M'_{\varphi(s)}$(the last equality holds because $\varphi(s)$ is a sort), so the models $\overline{M}$ and $\overline{M'} \upharpoonright_{\phi}$ coincide on sorts.

For each type $s \in \overrightarrow{S} \setminus S$, $\overline{M}_s = fun^M_s(M_s) = fun^M_s((M' \upharpoonright_{\phi(\varphi)})_s) = fun^M_s(M'_{\overrightarrow{\varphi}(s)})$. $(\overline{M'} \upharpoonright_{\phi})_s = \overline{M'}_{\overrightarrow{\varphi}(s)} = fun^{M'}_{\overrightarrow{\varphi}(s)}(M'_{\overrightarrow{\varphi}(s)})$. The equality holds by using remark 3.

For each constant symbol $\sigma \in F_s$, $\overline{M}_\sigma = fun^M_s(M_\sigma) = fun^{M' \upharpoonright_\varphi}_s(M'_{\varphi(\sigma)}) = fun^{M'}_{\varphi(s)}(M'_{\varphi(\sigma)}) = \overline{M'}_{\varphi(\sigma)} = (M' \upharpoonright_{\varphi})_\sigma$.

Let $h : M' \to N'$ be a $\varphi(S', F')$-model homomorphism. We must check that $\overline{h'} \upharpoonright_{\varphi}: \overline{M'} \upharpoonright_{\varphi} \to \overline{N'} \upharpoonright_{\varphi}$ and $\overline{h} : \overline{M' \upharpoonright_{\phi(\varphi)}} \to \overline{N' \upharpoonright_{\phi(\varphi)}}$ are equal, where we denote $h = h' \upharpoonright_{\phi(\varphi)}$.

By using the first part of the proof, we obtain that the two homomorphisms have the same domain and the same codomain.

Let $fun^{M'}_{\varphi(s)}(m)$ be an element of $(\overline{M'} \upharpoonright_{\varphi})_s = (\overline{M' \upharpoonright_{\phi(\varphi)}})_s$.

On one hand, $(\overline{h'} \upharpoonright_{\varphi})_s(fun^{M'}_{\varphi(s)}(m)) = \overline{h'}_{\varphi(s)}(fun^{M'}_{\varphi(s)}(m)) = fun^{N'}_{\varphi}(s)(h'_{\varphi(s)}(m))$.

On the other hand, $\overline{h}_s(fun^{M'}_{\varphi(s)}(m)) = fun^{N'}_{\varphi(s)}(h_s(m)) = fun^{N'}_{\varphi(s)}(h'_{\varphi(s)}(m))$.

It follows that $\overline{h'} \upharpoonright_{\varphi}$ and $\overline{h}$ are equal.

We obtained that β is indeed a natural transformation.

■

Counterexample 1 *If we consider type-derived signature morphisms, β is no longer a natural transformation.*

Let us consider the HNK-signature $\Sigma = (\{s\}, \emptyset)$ with only a sort symbol and the set of constant symbols empty for any type and the signature morphism $\varphi : \Sigma \to \Sigma$ defined by $\varphi(s) = s \to s$. According to the definition of ϕ, $\phi(\Sigma) = ((\overrightarrow{\{s\}}, \{apply_{t \to t'}\}_{t,t' \in \overrightarrow{\{s\}}}), \Gamma_\Sigma)$ and $\phi(\varphi)(t) = \varphi^{type}(t)$ for each type t, $\phi(\varphi)(apply_{t \to t'}) = apply_{\varphi^{type}(t) \to \varphi^{type}(t')}$.

$$\begin{array}{cccccc}
M' & Mod^{FOEQL}(\phi(\Sigma)) & \xrightarrow{\beta_\Sigma} & Mod^{HNK}(\Sigma) & & \overline{M'} \\
 & \Big\downarrow {\scriptstyle Mod^{FOEQL}(\phi(\varphi))} & & \Big\downarrow {\scriptstyle Mod^{HNK}(\varphi)} & & \\
M & Mod^{FOEQL}(\phi(\Sigma)) & \xrightarrow[\beta_\Sigma]{} & Mod^{HNK}(\Sigma) & & \overline{M}
\end{array}$$

Let M' be the following model in $Mod^{FOEQL}(\phi(\Sigma))$:

- $M'_s = \mathbb{N}$;

- $M'_{s \to s} = \{f\}$;

- $M'_{apply_{s\to s}}(f,x) = x$, *for any natural number* x.

We denote:

- $M = M' \upharpoonright_{\phi(\varphi)}$;
- $\overline{M'} = \beta_{(S',F')}(M')$;
- $\overline{M} = \beta_{(S,F)}(M)$;

Then:

- $\overline{M}_s$ =(*definition of* $\overline{M}$)M_s =(*definition of the reduct functor*)$M'_{\phi(\varphi)(s)}$ =(*definition of* $\phi(\varphi)$)$M'_{\varphi(s)} = M'_{s\to s'}$, *so* $\overline{M}_s = \{f\}$.
- $(\overline{M'} \upharpoonright_{\phi})_s$ =(*definition of the reduct functor*)$\overline{M'}_{\varphi(s)} = \overline{M'}_{s\to s'} = \{f : \mathbb{N} \to \mathbb{N} | f(x) = x\}$, *which is only isomorphic to* $\overline{M}_s$

  so the two models are not the same on the sort s.

Lemma 0.11 *For each HNK−signature* (S,F), *each F-term t of type s and each model* $M' \in |Mod^{FOEQL}(\phi(S,F))|$, $\overline{M'}_t = fun_s^{M'}(M'_{\alpha(t)})$.

Proof

Induction on the structure of terms.

Basic case: $t = \sigma$, with $\sigma \in F_s$. Then $\overline{M'}_\sigma = fun_s^{M'}(M'_\sigma)$, by using the definition of the interpretation of constant symbols in $\overline{M'}$.

General case: $t = t_1(t_2)$ and the inductive hypothesis holds for t_1 and t_2. $\overline{M'}_{t_1(t_2)} = \overline{M'}_{t_1}(\overline{M'}_{t_2}) =$ (by using the inductive hypothesis)$fun_{s\to s'}^{M'}(M'_{\alpha(t_1)})(fun_s^{M'}(M'_{\alpha(t_2)})) = fun_{s'}^{M'}(M'_{apply_{s\to s'}}(M'_{\alpha(t_1)}, M'_{\alpha(t_2)})) = fun_{s'}^{M'}(M'_{apply_{s\to s'}(\alpha(t_1),\alpha(t_2))}) = fun_{s'}^{M'}(M'_{\alpha(t_1(t_2))})$.

■

Proposition 0.12 *For any HNK-signature* (S,F), *any* $\phi(S,F)$*-model* M' *and any* (S,F)*-sentence e,*

$$M' \models^{FOEQL} \alpha_{(S,F)}(e) \iff \overline{M'} \models^{HNK} e$$

Proof:

The statement is proven by induction on the structure of sentence e.

For the basic case, let $e = (t = t')$. Then $\overline{M'} \models t = t' \iff \overline{M'}_t = \overline{M'}_{t'} \iff fun(M'_{\alpha(t)}) = fun(M'_{\alpha(t')}) \iff$ (because fun is isomorphism) $M'_{\alpha(t)} = M'_{\alpha(t')} \iff M' \models \alpha(t) = \alpha(t')$.

For the general case, the only non-trivial part is for universal quantification.

Assume that $e = (\forall X)\rho$ and the inductive hypothesis holds for ρ. We want to show that $M' \models (\forall X)\alpha_{(S,F)}(\rho)$ if and only if $\overline{M'} \models (\forall X)\rho$.

$$(S,F) \xrightarrow{\quad i_X \quad} (S, F \uplus X)$$

$$((\overrightarrow{S}, \overrightarrow{F}), \Gamma_{(S,F)}) \xrightarrow{i'_X} ((\overrightarrow{S}, \overrightarrow{F} \uplus X), \Gamma_{(S,F)}) \xrightarrow{i'_{\overrightarrow{X}}} ((\overrightarrow{S}, \overrightarrow{F \uplus X}), \Gamma_{(S,F \uplus X)})$$

Notice that:

1. $\overrightarrow{F} \uplus X = \overrightarrow{F \uplus X}$ so we can interchange the two of them. Indeed, $(\overrightarrow{F \uplus X})_{\to s} = (F \cup X)_s = F_s \cup X_s = (\overrightarrow{F})_{\to s} \cup X_s = (\overrightarrow{F} \uplus X)_{\to s}$, for any type $s \in \overrightarrow{S}$.

2. $\Gamma_{(S,F)} = \Gamma_{(S,F \cup X)}$, because the set of sorts is the same.

This implies that $i'_{\overrightarrow{X}}$ is actually the identity.

For showing $M' \models (\forall X)\alpha_{(S,F)}(\rho)$ implies $\overline{M'} \models (\forall X)\rho$, let N be an i_X-expansion of $\overline{M'}$. We define a $(\overrightarrow{S}, \overrightarrow{F \uplus X})$-model, N', as follows:

- for each sort $s \in \overrightarrow{S}$, $N'_s = M'_s$. Notice that $N_s = \overline{M'}_s = fun_s^{M'}(M_s'^{[]}) = fun_s^{N'}(N_s'^{[]})$;
- $N'_\sigma = M'_\sigma$;
- $N'_{apply_{s \to s'}}(f,x) = M'_{apply_{s \to s'}}(f,x)$;
- for any $x \in X_s$, there exists $a' \in M'_s$ such that $N_x = fun_s^{M'}(a')$. We define $N'_x = a'$.

With this definition, we notice that $\overline{N'} = N$ and N' is a i'_X-expansion of M'. The latter implies $N' \models \alpha(\rho)$ so by using the inductive hypothesis, $N \models \rho$.

For showing $M' \models (\forall X)\alpha_{(S,F)}(\rho)$ if $\overline{M'} \models (\forall X)\rho$, let N' be an i'_X-expansion of M'. We check that $\overline{N'}$ is an i_X-expansion of $\overline{M'}$.

- $\overline{N'}_s = fun_s^{N'}(N_s'^{[]}) = fun_s^{M'}(M_s'^{[]}) = \overline{M}_s$;
- $\overline{N'}_\sigma = fun_s^{N'}(N'_\sigma) = fun_s^{M'}(M'_\sigma) = \overline{M'}_\sigma$.

It follows that $\overline{N'} \models \rho$, so $N' \models \alpha(\rho)$.

■

Remark 4 *For each signature (S,F), $\beta_{(S,F)} : Mod^{FOEQL}(\phi(S,F)) \to Mod^{HNK}(S,F)$ is an equivalence of categories.*

Proof:
We define $\widetilde{\beta}_{(S,F)} : Mod^{HNK}(S,F) \to Mod^{FOEQL}(\phi(S,F))$ as follows:

- for each model $A \in Mod^{HNK}(S,F)$, we denote $\widetilde{A} = \widetilde{\beta}_{(S,F)}(A)$:

– for each type $s \in \overrightarrow{S}$, $\widetilde{A}_s = A_s$ (this means that the interpretation of each type in the model A loses the significance of function subset and is only regarded as a set);

– for each constant symbol $\sigma \in F_s$ of type $s \in \overrightarrow{S}$, $\widetilde{A}_\sigma = A_\sigma$;

– $\widetilde{A}_{apply_{s\to s'}}(f,x) = f(x)$, for each $f \in \widetilde{A}_{s\to s'}$ and each $x \in \widetilde{A}_s$ (this means we define the interpretation of the function $apply_{s\to s'}$ on a argument f that was a function in the model A by using the definition of f in the model A)

It is obvious that $\widetilde{A} \models \Gamma_{(S,F)}$, because the interpretations of types in $\widetilde{A}$ were function subspaces.

- for each $HNK-$model homomorphism $h : A \to B$, we define $\widetilde{h} = \widetilde{\beta}_{(S,F)}(h)$ by $\widetilde{h}_s(x) = h_s(x)$ for any type $s \in \overrightarrow{S}$ and any $x \in A_s$. We check that $\widetilde{h}$ is indeed a $FOEQL$-model homomorphism.

  For any constant symbol $\sigma \in F_s$, $\widetilde{h}_s(\widetilde{A}_\sigma) = h_s(A_\sigma) = B_\sigma = \widetilde{B}_\sigma$.

  For any types $s,s' \in \overrightarrow{S}$, any $f \in \widetilde{A}_{s\to s'}$ and any $x \in \widetilde{A}_s$, $\widetilde{h}_{s'}(\widetilde{A}_{apply_{s\to s'}}(f,x)) = h_{s'}(f(x)) = h_{s\to s'}(f)(h_s(x)) = \widetilde{B}_{apply_{s\to s'}}(\widetilde{h}_{s\to s'}(f), \widetilde{h}_s(x))$.

Fact 0.13 *$\widetilde{\beta}_{(S,F)}$ is a functor.*

We prove that $\beta_{(S,F)};\widetilde{\beta}_{(S,F)} \simeq 1_{Mod^{FOEQL}(\phi(S,F))}$ *and* $\widetilde{\beta}_{(S,F)};\beta_{(S,F)} = 1_{Mod^{HNK}(S,F)}$.

Let $M \in |Mod^{FOEQL}(\phi(S,F))|$. Notice that $\widetilde{\overline{M}}_s = \overline{M}_s = M_s = fun_s^M(M_s)$ if $s \in S$ and $\widetilde{\overline{M}}_{s\to s'} = \overline{M}_{s\to s'} = fun_{s\to s'}^M(M_{s\to s'}^{\square})$ if $s,s' \in \overrightarrow{S}$, so the function $g : M \to \widetilde{\overline{M}}$, $g_s(m) = fun_s^M(m)$ is bijective.

We check that g is a model homomorphism.

For any $\sigma \in F_s$, $g_s(M_\sigma) = fun_s^M(M_\sigma) = \overline{M}_\sigma = \widetilde{\overline{M}}_\sigma$.

For any $s,s' \in \overrightarrow{S}$, any $f \in M_{s\to s'}$ and any $x \in M_s$, $\widetilde{\overline{M}}_{apply_{s\to s'}}(g_{s\to s'}(f), g_s(x)) = \widetilde{\overline{M}}_{apply_{s\to s'}}(fun_{s\to s'}^M(f), fun_s^M(x)) = fun_{s\to s'}^M(f)(fun_s^M(x)) = fun_{s'}^M(M_{apply_{s\to s'}}(f,x)) = g_{s'}(M_{apply_{s\to s'}}(f,x))$.

It follows that g is an isomorphism of $FOEQL$ models.

Let $A \in |Mod^{HNK}(S,F)|$. We show that $\overline{\widetilde{A}} = A$ by induction over types.

Let $s \in S$. $\overline{\widetilde{A}}_s = \widetilde{A}_s = A_s$.

Let $s,s' \in \overrightarrow{S}$ such that $\overline{\widetilde{A}}_s = A_s$ and $\overline{\widetilde{A}}_{s'} = A_{s'}$. We know that $\overline{\widetilde{A}}_{s\to s'} \subseteq [\overline{\widetilde{A}}_s \to \overline{\widetilde{A}}_{s'}] = [A_s \to A_{s'}]$. Let $f \in \widetilde{A}_{s\to s'} = A_{s\to s'}$ and let $x \in \widetilde{A}_s$. By inductive hypothesis, $fun_s^{\widetilde{A}}(x) = x$. Then $fun_{s\to s'}^{\widetilde{A}}(f)(fun_s^{\widetilde{A}}(x)) = \widetilde{A}_{apply}(f,x) = f(x)$, so $fun_{s\to s'}^{\widetilde{A}}(f)$ is equal to f. The equality between $A_{s\to s'}$ and $\overline{\widetilde{A}}_{s\to s'}$ follows from the observation that for any $f \in A_{s\to s'}$ there exists $fun_{s\to s'}^{\widetilde{A}}(f) \in \overline{\widetilde{A}}_{s\to s'}$ such that $fun_{s\to s'}^{\widetilde{A}}(f) = f$.

Let $\sigma \in F_{\to s}$. $\overline{\widetilde{A}}_\sigma = fun_s^{\widetilde{A}}(\widetilde{A}_\sigma) = \widetilde{A}_\sigma = A_\sigma$.

The equality $\overline{\widetilde{h}} = h$ for h HNK-model homomorphisms is obvious.

■

0.5 Comorphism $FOEQL \to HNK$

We define an institution comorphism $(\phi, \alpha, \beta) : FOEQL \to HNK$. We can thus conclude that the first order equational logic (or even first order logic if we compose the comorphism defined here with the one presented as an example) is at most expresive as higher order logic with Henkin semantics.

- The functor $\phi : \mathbb{S}ign^{FOEQL} \to \mathbb{S}ign^{HNK}$ maps:
  - each $FOEQL$-signature (S,F) to the HNK-signature $(S,\overline{F})$, where $\overline{F}_{s_1 \to (s_2 \to \ldots (s_n \to s)..)} = F_{s_1 s_2 \ldots s_2 \to s}$ and $\overline{F}_x = \emptyset$ otherwise.
  - each $FOEQL$ signature morphism $\varphi : (S,F) \to (S',F')$ to the HNK signature morphism $\phi(\varphi) : (S,\overline{F}) \to (S',\overline{F'})$ such that $\phi(\varphi)(s) = \varphi(s)$ and associates to each constant symbol σ the constant symbol corresponding to $\varphi(\sigma)$. It is easy to see that $\phi(\varphi)$ is a HNK signature morphism.

- The natural transformation α is determined by the following mapping, also denoted α, on terms: each term $t = \sigma(t_1,..t_n)$ is assigned the Polish prefix translation of the term $\sigma(\alpha(t_1))(\alpha(t_2))..(\alpha(t_n))$.

- Let (S,F) be a $FOEQL$ signature. We define $\beta_{(S,F)} : Mod^{HNK}(\phi(S,F)) \to Mod^{FOEQL}(S,F)$ and we denote $\widehat{M} = \beta_{(S,F)}(M)$ and $\widehat{h} = \beta_{(S,F)}(h)$:
  - For any $(\phi(S,F))$-model M, we define $\widehat{M}$:
    1. $\widehat{M}_s = M_s$ for any sort $s \in S$;
    2. for each $\sigma \in F_{s_1 s_2..s_n \to s}$, $\widehat{M}_\sigma(x_1,..x_n) = M_\sigma(x_1)(x_2)..(x_n)$
  - For each HNK-model homomorphism $h : M \to N$, we define $\widehat{h} : \widehat{M} \to \widehat{N}$, $\widehat{h}_s = h_s$, for any sort $s \in S$.

    We check that $\widehat{h}$ is a $FOEQL$-model homomorphism. Let $\sigma \in F_{s_1 s_2..s_n \to s}$ and let $m_i \in M_{s_i}$, for any $i \in \{1,..,n\}$. Then $\widehat{h}(\widehat{M}_\sigma(m_1,..m_n)) = h(\widehat{M}_\sigma(m_1,..m_n)) = h(M_\sigma(m_1)..(m_n)) =$ $h(M_\sigma(m_1)..(m_{n-1}))(h(m_n)) = .. = h(M_\sigma)(h(m_1))..(h(m_n)) = N_\sigma(h(m_1))..(h(m_n)) =$ $\widehat{N}_\sigma(\widehat{h}(m_1),..,\widehat{h}(m_n))$.

Remark 5 β *is a natural transformation.*

Proof:

$$\begin{array}{cccccc} M' & Mod^{HNK}(\phi(S',F')) & \xrightarrow{\beta_{(S',F')}} & Mod^{FOEQL}(S',F') & & \widehat{M'} \\ & \downarrow{\scriptstyle Mod^{HNK}(\phi(\varphi))} & & \downarrow{\scriptstyle Mod^{FOEQL}(\varphi)} & & \\ M & Mod^{HNK}(\phi(S,F)) & \xrightarrow[\beta_{(S,F)}]{} & Mod^{FOEQL}(S,F) & & \widehat{M} \end{array}$$

Let $M' \in |Mod^{HNK}(\phi(S',F'))|$. We denote $M = Mod^{HNK}(\phi(\varphi))(M')$ and we check that $\widehat{M} = \widehat{M'} \upharpoonright_\varphi$.

Let $s \in S$. $\widehat{M}_s = M_s = M'_{\phi(\varphi(s))} = M'_{\varphi(s)} = \widehat{M'}_{\varphi(s)} = (\widehat{M'} \upharpoonright_\varphi)_s$.

Let $\sigma \in F_{s_1 s_2..s_n \to s}$. $\widehat{M}_\sigma(x_1,..,x_n) = M_\sigma(x_1)..(x_n) = M'_{\varphi(\sigma)}(x_1)...(x_n)$. $(\widehat{M'} \upharpoonright_\varphi)_\sigma(x_1,..,x_n) = \widehat{M'}_{\varphi(\sigma)}(x_1,..,x_n) =$ $M'_{\varphi(\sigma)}(x_1)...(x_n)$.

Let $h' : M' \to N'$ be a $(\phi(S', F'))-$model homomorphism. We denote $h = h' \upharpoonright_{\phi(\varphi)}$ and we check that $\widehat{h} = \widehat{h'} \upharpoonright_{\varphi}$.

The first part of the proof ensures us that the domain and the codomain of $\widehat{h}$ and $\widehat{h'} \upharpoonright_{\varphi}$ are the same. Let $s \in S$ and let $x \in \widehat{M}_s$. Then $\widehat{h}_s(x) = h_s(x) = h'_{\varphi(s)}(x)$. $(\widehat{h'} \upharpoonright_{\varphi})_s(x) = \widehat{h'}_{\varphi(s)}(x) = h'_{\varphi(s)}(x)$.

■

Lemma 0.14 *For any FOEQL signature (S, F), any $t \in T_F$ and any $\phi(S, F)$-model M, $M_{\alpha(t)} = \widehat{M}_t$.*

Proof:

Induction on the structure of terms.

Basic case: $t = \sigma$, with $\sigma \in F_{\to s}$. Then $\widehat{M}_\sigma = M_\sigma$ by using the definition of the interpretation of constant symbols in $\widehat{M}$.

General case: $t = \sigma(t_1, ..t_n)$ and the inductive hypothesis holds for each of $t_1, ..t_n$. Then $\widehat{M}_{\sigma(t_1,...,t_n)} = \widehat{M}_\sigma(\widehat{M}_{t_1}, .., \widehat{M}_{t_n})$ =(from the induction step and the definition of σ in $\widehat{M}$)$M_\sigma(M_{\alpha(t_1)})..(M_{\alpha(t_n)}) = M_{\sigma(\alpha(t_1))...(\alpha(t_n))} = M_{\alpha(\sigma(t_1,..t_n))}$.

■

Proposition 0.15 *For any FOEQL signature (S, F), any sentence $e \in Sen(S, F)$ and any $\phi(S, F)$ HNK-model M', $\widehat{M'} \models e \iff M' \models \alpha_{(S,F)}(e)$.*

Proof:

Induction on the structure of e:

Basic case: $e = (t = t')$. Then $\widehat{M'} \models t = t' \iff \widehat{M'}_t = \widehat{M'}_{t'} \iff M'_{\alpha(t)} = M'_{\alpha(t')} \iff M' \models \alpha(t) = \alpha(t')$.

For the general case, we only consider universal quantification. Notice that $(S, \overline{F} \cup X) = (S, \overline{F \cup X})$. Let us denote i_X the signature inclusion from (S, F) to $(S, F \cup X)$ and i'_X the signature inclusion from $(S, \overline{F})$ to $(S, \overline{F \cup X})$. We check that $M' \models (\forall X)\alpha(\rho) \iff \widehat{M'} \models (\forall X)\rho$.

For the left to right implication, let N be a i_X-expansion of $\widehat{M'}$. We define $N' \in |Mod^{HNK}(S, \bar{(}F \cup X))|$ as follows:

- $N'_s = M'_s$, for any sort $s \in S$;
- $N'_\sigma(x_1, .., x_n) = M'_\sigma(x_1, .., x_n)$, for any $\sigma \in F_{s_1..s_n \to s}$ and any $x_i \in N'_{s_i}$;
- for any $x \in X_s$, there exists $a \in M_s$ such that $N_x = a$. Then M'_x is interpreted as the function a.

Notice that with this definition N' is an i'_X-expansion of M' and therefore $N' \models \alpha(\rho)$. Since $\widehat{N'} = N$, we obtain $N \models \rho$.

For the right to left implication, let N' be a i'_X-expansion of M'. One can easily check that $\widehat{N'}$ is a i_X-expansion of $\widehat{M'}$ and therefore $\widehat{N'} \models \rho$, which further implies $N' \models \alpha(\rho)$.

■

0.6 Model-theoretic properties

0.6.1 Signature colimits

Proposition 0.16 *The category $\mathbb{S}ign^{HNK}$ has all small (co)limits.*

Proof:

For any set S, since $\mathbb{S}et$ has all small limits and colimits, the category $B(S) = Cat(\overrightarrow{S}, \mathbb{S}et)$ (where $\overrightarrow{S}$ is regarded as a discrete category) has small limits and colimits too.

Each function $f : S \to S'$ determines a functor $B(f) : \mathbb{C}at(\overrightarrow{S'}, Set) \to \mathbb{C}at(\overrightarrow{S}, Set)$ by composition to the left with $\overrightarrow{f}$ (i.e. for any function $F : \overrightarrow{S'} \to Set$, $B(f)(F) = \overrightarrow{f}; F$ and similarly for arrows).

This functor has a left adjoint mapping each F to F' such that $F'_{\to s'} = \biguplus\{F_{\to s} | \overrightarrow{f}(s) = s'\}$ and a right adjoint mapping each F to F'' such that $F''_{\to s'} = \Pi\{F_{\to s} | \overrightarrow{f}(s) = s'\}$.

Since all right adjoints preserve limits, we can apply the known result of existence of small limits/colimis in a Grothendieck category for the indexed category $B : \mathbb{S}et^{op} \to \mathbb{C}at$. It follows that $\mathbb{S}ign^{HNK}$, which is exactly the Grothendieck category over B, has small limits and colimits.

■

To emphasize the construction of limits/colimits in the concrete cases, we describe the way the pushouts of *HNK* signatures are built.

Let $\varphi_1 : (S, F) \to (S_1, F_1)$ and $\varphi_1 : (S, F) \to (S_1, F_1)$ be two *HNK* signature morphisms. We define a signature (S', F') and two signature morphisms $\theta_i : (S_i, F_i) \to (S', F')$ that will close the span forming a pushout.

We obtain the sort S' with the following pushout in *Set*:

$$\begin{array}{ccc} S & \xrightarrow{\varphi_1^{sort}} & S_1 \\ \downarrow{\scriptstyle \varphi_2^{sort}} & & \downarrow{\scriptstyle \theta_1^{sort}} \\ S_2 & \xrightarrow{\theta_2^{sort}} & S' \end{array}$$

The set S' is therefore defined as $S_1 \biguplus S_2 /_{(\varphi_1(s), \varphi_2(s))_{s \in S}}$.

By considering the extensions of the signature morphisms to types, we obtain a commutative square that is <u>not</u> a pushout (because $\overrightarrow{S'}$ contains types like $s_1 \to s_2$, with $s_i \in S_i$).

$$\begin{array}{ccc} \overrightarrow{S} & \xrightarrow{\varphi_1^{type}} & \overrightarrow{S_1} \\ \downarrow{\scriptstyle \varphi_2^{type}} & & \downarrow{\scriptstyle \theta_1^{type}} \\ \overrightarrow{S_2} & \xrightarrow{\theta_2^{type}} & \overrightarrow{S'} \end{array}$$

Let us denote S'' the pushout of $\overrightarrow{S_1} \overset{\varphi_1^{type}}{\leftarrow} \overrightarrow{S} \overset{\varphi_2^{type}}{\rightarrow} \overrightarrow{S_2}$ in *Set* and notice that it is a subset of $\overrightarrow{S'}$.

We only have to define F'. For any type $s' \in \overrightarrow{S'}$, if s' is not in S'' then $F'_{s'} = \emptyset$, otherwise there exists $s_1 \in \overrightarrow{S_1}$ or $s_2 \in \overrightarrow{S_2}$ or the both of them such that $\theta_i(s_i) = s'$. If only one of them exists, $F'_{s'} = (F_i)_{s_i}$; if both of them exist, $F'_{s'} = (F_1)_{s_1} \biguplus (F_2)_{s_2} /_{\biguplus_{s \in \overrightarrow{S}, \varphi^{type}(s) = s'} (\varphi_1^{op}(\sigma), \varphi_2^{op}(\sigma))_{\sigma \in F_s}}$.

Routine calculation show that we obtained indeed a signature pushout.

0.6.2 Model amalgamation

Definition 9 *In any institution, a commuting square of signatures*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

is an amalgamation square if and only if for each Σ_1-model M_1 and each Σ_2-model M_2 such that $M_1 \upharpoonright_{\varphi_1} = M_2 \upharpoonright_{\varphi_2}$, there exists a unique Σ'-model M', called the amalgamation of M_1 and M_2 such that $M' \upharpoonright_{\theta_i} = M_i$. If M' is not unique, we say the square has the weak amalgamation property.

Definition 10 *An institution has the model amalgamation property if every pushout of signatures is an amalgamation square.*

Counterexample 2 *HNK does not have the strong amalgamation property.*

Let us consider the following signature pushout:

$$\begin{array}{ccc} (\{s\},\emptyset) & \hookrightarrow & (\{s,s_1\},\emptyset) \\ \downarrow & & \downarrow \\ (\{s,s_2\},\emptyset) & \hookrightarrow & (\{s,s_1,s_2\},\emptyset) \end{array}$$

Let A_1 be the following $\Sigma_1 = (\{s,s_1\},\emptyset)$-model:

- $(A_1)_s$ *and* $(A_1)_{s_1}$ *are interpreted as arbitrary non-empty sets;*
- *all the types are interpreted as the empty set.*

and let A_2 be the following $\Sigma_2 = (\{s,s_2\},\emptyset)$-model:

- $(A_2)_s$ *and* $(A_2)_{s_2}$ *are interpreted as sets, such that* $(A_1)_s = (A_2)_s$*;*
- *all the types are interpreted as the empty set.*

The condition that the models interpret s as the same set and all types built with s as the empty set implies that the models are the same when reduced to the signature $(\{s\},\emptyset)$.

We consider two $(\{s,s_1,s_2\},\emptyset)$-models, M_1 and M_2:

- $(M_1)_s = (M_2)_s = (A_1)_s = (A_2)_s$*;*
- $(M_1)_{s_1} = (M_2)_{s_1} = (A_1)_{s_1}$*;*
- $(M_1)_{s_2} = (M_2)_{s_2} = (A_2)_{s_2}$*;*
- $(M_1)_{s_1 \to s_2} = \emptyset$*;*

- $(M_2)_{s_1 \to s_2} = [(M_2)_{s_1} \to (M_2)_{s_2}]$;
- *the other types are interpreted as the empty set.*

Notice that $M_i \upharpoonright_{\Sigma_i} = A_i$*, for* $i \in \{1, 2\}$ *so the amalgamation of* A_1 *and* A_2 *is not unique.*

Proposition 0.17 *HNK has the weak amalgamation property.*

Proof:

Let us consider the following signature pushout:

$$\begin{array}{ccc} (S, F) & \xrightarrow{\varphi_1} & (S_1, F_1) \\ \downarrow \varphi_2 & & \downarrow \theta_1 \\ (S_2, F_2) & \xrightarrow{\theta_2} & (S', F') \end{array}$$

and let M_1 be a (S_1, F_1)-model, M_2 a (S_2, F_2)-model such that $M_1 \upharpoonright_{\varphi_1} = M_2 \upharpoonright_{\varphi_2}$. We show that there exists a (S', F')-model M' such that $M' \upharpoonright_{\theta_i} = M_i$.

The model M' interprets:

- the sorts: for each sort $s' \in S'$, if there exists a sort $s_1 \in S_1$ such that $\theta_1(s_1) = s'$, the set $M'_{s'}$ will be defined as $(M_1)_{s_1}$, otherwise it will be defined as $(M_2)_{s_2}$. If the sort s' is the image of sort $s \in S$, it makes no difference if we define it as $(M_1)_{\varphi_1(s)}$ or $(M_2)_{\varphi_2(s)}$ because the sets are equal.
- the types: for each type $s' \in \overrightarrow{S'}$, if s' is not in the set S'' defined in the construction of the signature pushout, $M'_{s'}$ may be interpreted in any way and we choose to define it as the empty set. Otherwise, the definition of $M'_{s'}$ is similar to the one for sorts.
- the constant symbols: for each constant symbol $\sigma' \in F'$, if there exists an constant symbol $\sigma_1 \in F_1$ such that $\theta_1(\sigma_1) = \sigma'$, we define $M'_{\sigma'} = (M_1)_{\sigma_1}$, otherwise it will be defined as $(M_2)_{\sigma_2}$. The definition is consistent because of the way we constructed the signature pushout.

It follows easily from the definition of M' that $M' \upharpoonright_{\theta_i} = M_i$.

■

0.6.3 Initial model for signatures

Counterexample 3 *Not all HNK-signatures have an initial model.*

Let $S = \{s\}$ *and* $F_{s \to s} = \{f, g\}$.

We build the following model, denoted 0*:*

- $0_s = \emptyset$;
- $0_{s \to s} = \{1_\emptyset\}$;
- $0_f = 0_g = 1_\emptyset$.

It is obvious that 0 *is not the initial model of* (S,F) *- we cannot define a function from* 0 *to a model* M *with* $M_f \neq M_g$ *such that the morphism condition holds.*

Assume that a model M *is the initial model of* (S,F)*. Then there exists* $h : M \to 0$*. Because the codomain of* h_s *is empty, the set* M_s *must be empty too. It follows that* $M_f = M_g = 1_{\emptyset}$*, which cannot be true for the initial model of* (S,F)*.*

Proposition 0.18 *If a signature* (S,F) *has at least a constant operation symbol of each type, then* $Mod^{HNK}(S,F)$ *has an initial model.*

Proof:

We show that $Mod^{FOEQL}(\phi(S,F))$ has an initial model.

Let 0 be the following model:

- 0_s is the set of terms of type s, for each type $s \in \overrightarrow{S}$;
- $0_\sigma = \sigma$, for each constant symbol $\sigma \in F_s$, with $s \in S$;
- $0_\sigma(t) = \sigma(t)$, for each constant symbol $\sigma \in F_{s\to s'}$ and each term t of type s;
- $0_{apply_{s\to s'}}(t,t') = apply_{s\to s'}(t,t')$.

Notice that for any $f,g \in 0_{s\to s'}$, $0 \models (\forall x)apply_{s\to s'}(f,x) = apply_{s\to s'}(g,x)$ because the equality $0_{apply_{s\to s'}}(f,x) = 0_{apply_{s\to s'}}(g,x)$ is equivalent to the syntactical equality between $apply_{s\to s'}(f,x)$ and $apply_{s\to s'}(g,x)$, which is false and a term that can be substituted for x always exists.

Let M be a (S,F)-model. We define $h : 0 \to M$, $h_s(t) = M_t$ for any type $s \in \overrightarrow{S}$ and any term t of type s.

We check that h is a $FOEQL$-model homomorphism, by structural induction on terms. For the basic case, let $\sigma \in F_s$. Then $h_s(\sigma) = M_\sigma$. For the induction step, $h_{s\to s'}(0_{apply_{s\to s'}}(t,t')) = h_{s\to s'}(apply_{s\to s'}(t,t')) = M_{apply_{s\to s'}(t,t')} = M_{apply_{s\to s'}}(M_t, M'_t) = M_{apply_{s\to s'}}(h(t), h(t'))$.

We check the uniqueness of the morphism h. Assume there exists another morphism $h' : 0 \to M$. We show that $h'(x) = h(x)$ for any term x by structural induction on terms. If $\sigma \in F_s$, $h'(\sigma) =$ (morphism condition for h')$M_\sigma = h(\sigma)$. For the induction step, $h'(0_{apply_{s\to s'}}(t,t')) = M_{apply_{s\to s'}}(h'(t), h'(t')) =$(inductive hypothesis)$M_{apply_{s\to s'}}(h(t), h(t')) = h(0_{apply_{s\to s'}}(t,t'))$.

■

0.6.4 Representable and quasi-representable signature morphisms

The institutional notion of *representable* signature morphisms is an abstract concept meant to capture the phenomena of quantification over (sets of) first-order variables. The notion starts from the fact that semantics of quantification in first-order-like logics can be given in terms of signature extensions: $M \models_{(S,F,P)} (\forall X)e$ ($M \models_{(S,F,P)} (\exists X)e$) iff $M' \models_{(S,F\uplus X,P)} e$ for each (for some) $(S,F\uplus X,P)$-expansion M' of M. Thus, in order to reach first-order quantification institutionally, one needs to define somehow what "injective signature morphism that only adds constant symbols" (such as $\iota : (S,F,P) \to (S,F\uplus X,P)$) means.

Definition 11 *In an institution I, a signature morphism* $\varphi : I \to I'$ *is:*

1. *quasi-representable[8] if for any* Σ'*-model* A' *the canonical functor determined by the reduct is an isomorphism of categories between* $A'/Mod(\Sigma')$ *and* $A' \upharpoonright_{\varphi} /Mod(\Sigma)$*. This means that each model homomorphism* $h : A' \upharpoonright_{\varphi} \to B$ *admits a unique expansion* $h' : A' \to B'$*.*

2. *representable[8] if there exists a* Σ*-model* M_{φ}*, called the representation of* φ *and an isomorphism of categories* $i_{\varphi} : Mod(\Sigma') \to M_{\varphi}/Mod(\Sigma)$*, such that* $i_{\varphi};forgetful = Mod(\varphi)$*, where forgetful is the usual functor mapping each arrow to its codomain.*

Fact 0.19 *[8] In* FOL*, each signature extension with constants is representable.*

The two concepts are related by the following result:

Fact 0.20 *[8] A signature morphism* $\varphi : \Sigma \to \Sigma'$ *it is representable if and only if is quasi-representable and* $Mod(\Sigma')$ *has an initial model.*

Proposition 0.21 *All* HNK*-signature extensions with constants symbols are quasi-representable.*

Proof:

Let $\varphi : (S,F) \to (S, F \uplus F')$ be a HNK-signature inclusion and let A' be a $(S, F \uplus F')$-model. We denote $A = A' \upharpoonright_{\varphi}$.

We must define an isomorphism of categories between $A'/Mod(S, F \uplus F')$ and $A/Mod(S,F)$.

Let $i_{A',(S,F)} : A'/Mod(S, F \uplus F') \to A/Mod(S,F)$ be the following functor:

- for any object $h' : A' \to B'$ of $A'/Mod(S, F \uplus F')$, we define $i_{A',(S,F)}(h') = h' \upharpoonright_{\varphi}$,which is an object in $A/Mod(S,F)$;
- for any arrow $g' : (h_1 : A' \to B_1) \to (h_2 : A' \to B_2)$, we define $i_{A',(S,F)}(g') = g' \upharpoonright_{\varphi}$, which is an arrow in the comma category $A/Mod(S,F)$.

Let $i'_{A',(S,F)} : A/Mod(S,F) \to A'/Mod(S, F \uplus F')$ be the functor defined as follows:

- for any (S,F)-model homomorphism $h : A \to B$, we define a $(S, F \uplus F')$-model B' as follows:
  - $B' \upharpoonright_{\varphi} = B$;
  - $B'_{\sigma} = h_s(A'_{\sigma})$, for each $\sigma \in F'_s$.

  We denote $h' = i^{-1}_{A',(S,F)}(h) : A' \to B'$ which is defined by $h'_s(a) = h_s(a)$, for each $a \in A'_s$ and each $s \in \overrightarrow{S}$. Notice that h' is a model homomorphism between A' and B' by using that h is a model homomorphism and the definition of the model B' on the constant symbols in F'.

- for any arrow $g : (h_1 : A \to B_1) \to (h_2 : A \to B_2)$, we define $g' : (h'_1 : A' \to B'_1) \to (h'_2 : A' \to B'_2)$ by $g'_s(b) = g_s(b)$, for any $b \in (B'_1)_s$. It suffices to prove the homomorphism condition for constant symbols in F': $g'((B'_1)_\sigma) = g'(h'_1(A'_\sigma)) = g(h_1(A'_\sigma)) = h_2(A'_\sigma) = h'_2(A'_\sigma) = (B'_2)_\sigma$, for any $\sigma \in F'_{\to s}$.

  We show that g' is indeed an arrow in the comma category: $(h'_1; g')_s(a) = g'_s((h'_1)_s(a)) = g'_s((h_1)_s(a)) = g_s((h_1)_s(a)) = (h_2)_s(a) = (h'_2)_s(a)$, for any $a \in A'_s$.

It is easy to check that $i_{A',(S,F)}$ is an isomorphism of categories. ■

Remark 6 *If $(S, F \uplus F')$ has an initial model, then $\varphi : (S,F) \to (S, F \uplus F')$ is representable.*

0.6.5 Substitutions

Given a FOL signature (S,F,P) and two sets of new constants, called *first order variables* X and Y, a *first order* (S,F,P)*-substitution* from X to Y consists of a mapping $\psi : X \to T_F(Y)$ of the variables X with F-terms over Y.

On the semantics side, each (S,F,P)-substitution $\psi : X \to T_F(Y)$ determines a functor

$$Mod(\psi) : Mod(S, F \uplus Y, P) \to Mod(S, F \uplus X, P)$$

defined by

- $Mod(\psi)(M)_x = M_x$ for each sort $s \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and

- $Mod(\psi)(M)_x = M_{\psi(x)}$, i.e. the evaluation of the term $\psi(x)$ in M, for each $x \in X$.

On the syntax side, ψ determines a sentence translation function

$$Sen(\psi) : Sen(S, F \uplus X, P) \to Sen(S, F \uplus Y, P)$$

which in essence replaces all symbols from X with the corresponding $(F \uplus Y)$-terms according to ψ. This can be formally defined as follows:

- $Sen(\psi)(t = t')$ is defined as $\overline{\psi}(t) = \overline{\psi}(t')$ for each $(S, F \uplus X, P)$-equation $t = t'$, where $\overline{\psi} : T_F(X) \to T_F(Y)$ is the unique extension of ψ to an F-morphism ($\overline{\psi}$ is replacing variables $x \in X$ with $\psi(x)$ in each $F \cup X$-term t).

- $Sen(\psi)(\pi(t_1, \ldots, t_n))$ is defined as $\pi(\overline{\psi}(t_1), \ldots, \overline{\psi}(t_n))$ for each $(S, F \uplus X, P)$-relational atom $\pi(t_1, \ldots, t_n)$.

- $Sen(\psi)(\rho_1 \wedge \rho_2)$ is defined as $Sen(\psi)(\rho_1) \wedge \mathsf{Sen}(\psi)(\rho_2)$ for each conjunction $\rho_1 \wedge \rho_2$ of $(S, F \uplus X, P)$-sentences, and similarly for the case of any other logical connectives.

- $Sen(\psi)((\forall Z)\rho)$ is defined as $(\forall Z)\mathsf{Sen}(\psi_Z)(\rho)$ for each $(S, F \uplus X \uplus Z, P)$-sentence ρ, where ψ_Z is the trivial extension of ψ to an $(S, F \uplus Z, P)$-substitution.

Note that we have extended the notations used for the models functor Mod and for the sentence functor Sen from the signatures to the first order substitutions. This notational extension is justified by the satisfaction condition given by the Proposition 0.22 below.

Proposition 0.22 *[10] For each* **FOL***-signature (S,F,P) and each (S,F,P)-substitution $\psi : X \to T_F(Y)$,*

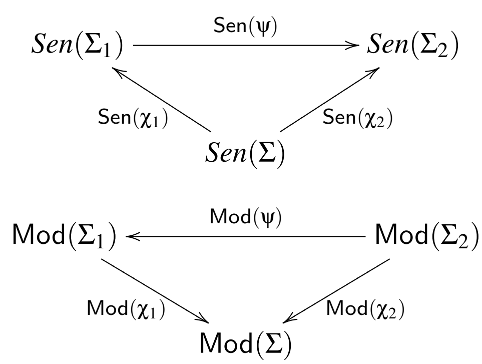
$$Mod(\psi)(M) \models \rho \text{ if and only if } M \models \mathsf{Sen}(\psi)(\rho)$$

for each $(S, F \uplus Y, P)$*-model* M *and each* $(S, F \uplus X, P)$*-sentence* ρ.

The satisfaction condition property expressed in Proposition 0.22 permits the definition of a general concept of substitution by abstracting

- **FOL** signatures (S, F, P) to signatures $\mathbb{S}$ in arbitrary institutions, and
- sets of first order variables X for (S, F, P) to signature morphisms $\Sigma \to \Sigma_1$.

Definition 12 *In an institution* I*, given a signature* Σ *and two signature morphisms* $\chi_1 : \Sigma \to \Sigma_1$ *and* $\chi_2 : \Sigma \to \Sigma_2$*, a general* Σ*-substitution [10]* $\psi : \chi_1 \to \chi_2$ *is a pair* $(Mod(\psi), Sen(\psi))$*, where* $Mod(\psi) : Mod(\Sigma_2) \to Mod(\Sigma_1)$ *is a functor and* $Sen(\psi) : Sen(\Sigma_1) \to Sen(\Sigma_2)$ *is a function, both preserving* Σ *i.e. the following diagrams commute:*



and the following satisfaction condition for substitutions holds:

$$Mod(\psi)(M) \models e \iff M \models Sen(\psi)(e)$$

for each Σ_2*-model* M *and each* Σ_1*-sentence* e*.*

Proposition 0.23 *Let* $\varphi_1 : (S, F) \to (S \cup S_1, F \cup F_1)$ *and* $\varphi_2 : (S, F) \to (S \cup S_2, F \cup F_2)$ *be two signature inclusions in HNK. A mapping* $\psi^{st} : S_1 \to \overrightarrow{S \cup S_2}$ *and a family of functions* $\{\psi^{op}_s : (F_1)_s \to (T_{F \cup F_2})_{\psi^{type}(s)} | s \in \overrightarrow{S \cup S_1}\}$ *determines a substitution* $\psi : \varphi_1 \to \varphi_2$*.*

Proof:

We can extend the domain of ψ^{st} to $S \cup S_1$ by defining $\psi^{st}(s) = s$, for any $s \in S$ and similarly, the domain of ψ^{op}_s to $F \cup F_1$ by defining $\psi^{op}_s(\sigma) = \sigma$, for any constant symbol $\sigma \in F_s$.

1. *The function Sen*

   Let us define a function $\psi : T_{F \cup F_1} \to T_{F \cup F_2}$, by induction on the structure of terms:

   - $\psi(\sigma) = \psi^{op}(\sigma)$, for any constant symbol $\sigma \in F \cup F_1$;
   - $\psi(t(t')) = \psi(t)(\psi(t'))$.

By extending this function to sentences we obtain a function $Sen(\psi) : Sen(\Sigma_1) \to Sen(\Sigma_2)$. Notice that $Sen(\psi)((\forall X)e) = (\forall X^{\Psi})\psi(e)$, where any variable from X of sort s becomes a variable in X^{Ψ} of sort $\psi(s)$.

Lemma 0.24 *$Sen(\psi)$ preserves (S,F).*

Proof of lemma:

Obvious, because $Sen(\varphi_2)(e) = e = Sen(\psi)(Sen(\varphi_1)(e))$, for any (S,F)-sentence e.

■

2. *The functor Mod*

Let M' be an $(S \cup S_2, F \cup F_2)$-model. We denote $Mod(\psi)(M') = M$, where we define:

- $M_s = M'_{\psi^{type}(s)}$, for any type $s \in \overrightarrow{S \cup S_1}$ - notice that this implies $M_s = M'_s$ for any type $s \in \overrightarrow{S}$ and also that $M_{s \to s'} = M'_{\psi(s) \to \psi(s')} \subseteq [M'_{\psi(s)} \to M'_{\psi(s')}] = [M_s \to M'_{s'}]$, so M interprets the types correctly ;
- $M_\sigma = M'_{\psi_s^{op}(\sigma)}$, for any constant symbol $\sigma \in (F \cup F_1)_s$ - the definition is correct because $M'_{\psi_s^{op}(\sigma)}$ is an element of $M'_{\psi(s)} = M_s$.

Let $h' : M' \to N'$ be a $(S \cup S_2, F \cup F_2)$-model homomorphism. We denote $Mod(\psi)(h') = h : M \to N$, where $h_s(x) = h'_{\psi(s)}(x)$.

Let us check that h is indeed a model homomorphism.

Let $\sigma \in (F \cup F_1)_s$. Then $h_s(M_\sigma) = h'_{\psi(s)}(M'_{\psi(\sigma)})$ =(because h' is a $(S \cup S_2, F \cup F_2)$-model homomorphism) $N'_{\psi(\sigma)} = N_\sigma$.

For any type $s \to s' \in \overrightarrow{S \cup S_1}$, any $f \in M_{s \to s'}$ and any $x \in M_s$, $h_{s \to s'}(f)(h_s(x)) = h'_{\psi(s) \to \psi(s')}(f)(h'_{\psi(s)}(x)) = h'_{\psi(s')}(f(x)) = h_{s'}(f(x))$.

Lemma 0.25 *$Mod(\psi)$ preserves (S,F).*

Proof of lemma:

Let M' be a $(S \cup S_2, F \cup F_2)$-model. We check that $M \upharpoonright_{\varphi_1} = M' \upharpoonright_{\varphi_2}$.

For any type $s \in \overrightarrow{S}$, $(M \upharpoonright_{\varphi_1})_s = M_s = M'_{\psi(s)}$ =(because $s \in S$) $M'_s = (M' \upharpoonright_{\varphi_2})_s$.

For any constant symbol $\sigma \in F_s$, $(M \upharpoonright_{\varphi_1})_\sigma = M_\sigma = M'_{\psi(\sigma)}$ =(because $\sigma \in F$) $M'_\sigma = (M' \upharpoonright_{\varphi_2})_\sigma$.

Let $h' : M' \to N'$ be a $(S \cup S_2, F \cup F_2)$-model homomorphism. We check that $h \upharpoonright_{\varphi_1} = h' \upharpoonright_{\varphi_2}$. The first part of the proof ensures us that they have the same domain and same codomain.

For any type $s \in \overrightarrow{S}$ and any $x \in M_s$, $(h \upharpoonright_{\varphi_1})_s(x) = h_s(x) = h'_s(x) = (h' \upharpoonright_{\varphi_2})_s(x)$.

■

3. *The satisfaction condition for substitutions*

Proposition 0.26 *We prove that for any model $M' \in |Mod(S \cup S_2, F \cup F_2)|$ and any sentence $e \in Sen(S \cup S_1, F \cup F_1)$, $M' \models Sen(\psi)(e) \iff Mod(\psi)(M') \models e$.*

Proof:

We begin with the following statement, that can be easily proved by induction on the structure of terms: for any $(S \cup S_1, F \cup F_1)$-term t and any $(S \cup S_2, F \cup F_2)$-model M', $(Mod(\psi)(M'))_t = M'_{\psi(t)}$.

The proof for the satisfaction condition is then made by induction on the structure of sentence e. We denote $(Mod(\psi)(M')) = M$.

For the basic case, let $e = (t = t')$. Then $M \models t = t' \iff M_t = M_{t'} \iff M'_{\psi(t)} = M'_{\psi(t')} \iff M' \models \psi(t) = \psi(t')$.

For the general case, the only non-trivial subcase is for universal quantification, for which we notice that the i_X-expansions of M are in bijective correspondence with the i_{X^ψ}-expansions of M' .

$$\begin{array}{ccc} (S \cup S_1, F \cup F_1) & \xrightarrow{\psi} & (S \cup S_2, F \cup F_2) \\ \big\downarrow i_X & & \big\downarrow i_{X^\psi} \\ (S \cup S_1, F \cup F_1 \cup X) & \xrightarrow{\psi_X} & (S \cup S_2, F \cup F_2 \cup X^\psi) \end{array}$$

■

This completes the proof of Proposition 0.23.

■

0.6.6 Filtered products

Filtered products in institutions.

Definition 13 *A filter F over a set I is a set $F \subseteq \mathcal{P}(I)$ such that*

- *$I \in F$;*
- *$X \cap Y \in F$ if X and Y are in F;*
- *$Y \in F$ if $X \subseteq Y$ and $X \in F$.*

A filter F is called ultrafilter when $X \in F \iff I \setminus X \in F$, for any $X \in \mathcal{P}(I)$.

Definition 14 *Let F be a filter over I and let $\{A_i\}_{i \in I}$ be a family of objects in a category $\mathbb{C}$ with small products. Let $A_F : F \to \mathbb{C}$ be the functor mapping each set $J \in F$ to $\Pi_{i \in J} A_i$ and each inclusion $J \subseteq J'$ to the canonical projection $p_{J'J} : \Pi_{i \in J'} A_i \to \Pi_{i \in J} A_i$. A filtered product of $\{A_i\}_{i \in I}$ modulo F [8] is a colimit $\mu : A_F \to \Pi_F A_i$ of the functor A_F.*

If F is an ultrafilter, a filtered product modulo F is called ultrafilter.

Proposition 0.27 *For any HNK-signature (S,F), $Mod^{HNK}(S,F)$ has direct products.*

Proof:

Let $\{A_i\}_{i\in I}$ be a family of (S,F)-models and let us consider the institution comorphism $(\phi,\alpha,\beta) : HNK \to FOEQL^p$ defined in section 0.4. We check that the product P of the family $\{\widetilde{A_i}\}_{i\in I}$ in the category $Mod^{FOEQL}(\overrightarrow{S},\overrightarrow{F})$ satisfies the axioms in $\Gamma_{(S,F)}$. If so, then $\Pi_{i\in I}A_i = \overline{P}$.

Let $f,g \in P_{s\to s'}$ such that $(\forall x \in P_s)P_{apply_{s\to s'}}(f,x) = P_{apply_{s\to s'}}(g,x)$. If $P_s = \emptyset$, then $P_{s\to s'}$ has only one element and therefore $f = g$. If $P_s \neq \emptyset$, $(\forall x \in P_s)P_{apply_{s\to s'}}(f,x) = P_{apply_{s\to s'}}(g,x) \iff (\forall i \in I)(\forall x \in (\widetilde{A_i})_s)(\widetilde{A_i})_{apply_{s\to s'}}(f_i,x) = (\widetilde{A_i})_{apply_{s\to s'}}(g_i,x) \iff (\forall i \in I) f_i = g_i \iff f = g$.

■

Proposition 0.28 *For any HNK-signature (S,F), any family of (S,F)-models $(A_i)_{i\in I}$ and any ultrafilter U over I, the ultraproduct $\Pi_U A_i$ always exists.*

Proof:

Let $A_U : U \to Mod^{HNK}(S,F)$, $A_U(J) = \Pi_J A_i$ and $A_U(J \subset J') = p_{J'J} : \Pi_{J'}A_i \to \Pi_J A_i$. We want to show that the functor A_U has colimits.

Let us consider the institution comorphism $(\phi,\alpha,\beta) : HNK \to FOEQL^p$ defined in section 0.4. We know that the family $B_i = \widetilde{A}_{i i\in I}$ has an ultraproduct B in the category $Mod(\overrightarrow{S},\overrightarrow{F})$. Moreover, since all the sentences in $\Gamma_{(S,F)}$ are preserved by ultraproducts (because $FOEQL$ is a Łoś institution), B is a $\phi(S,F)$-model. It follows that A_U has a colimit $A = \overline{B}$, since the direct products $\Pi_J A_i$ are equal to $\overline{\Pi_J B_i}$, the functor B_U has colimits and $\beta_{(S,F)}$ is an equivalence of categories.

■

Directed colimits of HNK models

Let (S,F) be a HNK signature, $(I,\leq)$ a directed set and $A : (I,\leq) \to Mod^{HNK}(S,F)$ be a functor. Let $(\{\mu_i\}_{i\in I},B)$ be the directed colimit of $A;\widetilde{\beta}_{(S,F)};U$ where U is the forgetful functor from $FOEQL$ presentations to signatures. When $B \models \Gamma_{(S,F)}$, because $\beta_{(S,F)}$ is an equivalence of categories, the colimit of A is $(\{\overline{\mu_i}\}_{i\in I},\overline{B})$.

Let us denote $h^{i,j} = (A;\widetilde{\beta}_{(S,F)})(i \leq j)$ and $B^i = (A;\widetilde{\beta}_{(S,F)})(i)$. We recall the way the directed colimit B is defined in $Mod^{FOEQL}(\overrightarrow{S},\overrightarrow{F})$:

- for each $s \in \overrightarrow{S}$, the set $B_s = \biguplus_{i\in I}(B^i_s)/_{\equiv_s}$, where for any $i,j \in I$, any $a \in B^i_s$, $b \in B^j_s$, $a \equiv_s b$ if and only if there exists $k \in I$ such that $i,j \leq k$ and $h^{i,k}(a) = h^{j,k}(b)$. We denote $a/_{\equiv}$ the equivalence class of an element a.
- for any constant symbol $\sigma \in F_s$, $B_\sigma = B^i_\sigma/_{\equiv}$

- for any types $s, s' \in \overrightarrow{S}$, $B_{apply_{s\to s'}}(f/_{\equiv}, x/_{\equiv}) = (B^k_{apply_{s\to s'}}(h^{i,k}(f), h^{j,k}(x))/_{\equiv}$ if $f \in B^i_{s\to s'}$, $x \in B^j_s$ and $i, j \leq k$.

Counterexample 4 *Directed colimits of HNK-models do not always exist.*

Proof:

Let (S,F) be a HNK-signature where $S = \{s\}$ and $F_{\to x} = \{f, g\}$ for all $x \in \overrightarrow{S}$, and let $I = (\mathbb{N}, \leq)$ which is a directed set.

For any $i \in \mathbb{N}$, we define A^i as the following model:

- $A^i_s = \{n \in \mathbb{N} | n \geq i\}$;
- $A^i_{s\to s} = \{f^i, g^i\}$, where $f^i(x) = x$ for any $x \in A^i_s$ and

$$g^i(x) = \begin{cases} x-1 & \text{if } x > i \\ i & \text{if } x = i \end{cases}$$

- $A^i_f = f^i$, $A^i_g = g^i$.

For any $i \in \mathbb{N}$, we define $a^{i,i+1} : A^i \to A^{i+1}$:

- $$a^{i,i+1}_s(x) = \begin{cases} x & \text{if } x > i \\ i+1 & \text{if } x = i \end{cases}$$
- $a^{i,i+1}_{s\to s}(f^i) = f^{i+1}$ and $a^{i,i+1}_{s\to s}(g^i) = g^{i+1}$.

We check that $a^{i,i+1}$ is a model homomorphism.

For any $x \in A^i_s$, $a^{i,i+1}(f^i)a^{i,i+1}(x) = f^{i+1}(a^{i,i+1}(x)) = a^{i,i+1}(x) = a^{i,i+1}(f^i(x))$, so the homomorphism condition holds for f^i.

We check that $a^{i,i+1}(g^i)a^{i,i+1}(i) = a^{i,i+1}(g^i(i))$. On one hand, $a^{i,i+1}(g^i)a^{i,i+1}(i) = g^{i+1}(i+1) = i+1$ and on the other hand $a^{i,i+1}(g^i(i)) = a^{i,i+1}(i) = i+1$.

We check that $a^{i,i+1}(g^i)a^{i,i+1}(i+1) = a^{i,i+1}(g^i(i+1))$. On one hand, $a^{i,i+1}(g^i)a^{i,i+1}(i+1) = g^{i+1}(i+1) = i+1$ and on the other hand $a^{i,i+1}(g^i(i+1)) = a^{i,i+1}(i) = i+1$.

We check that $a^{i,i+1}(g^i)a^{i,i+1}(x) = a^{i,i+1}(g^i(x))$ for any $x > i+1$. On one hand, $a^{i,i+1}(g^i)a^{i,i+1}(x) =$(because $x > i$)$g^{i+1}(x) =$(because $x > i$)$x-1$ and on the other hand $a^{i,i+1}(g^i(x)) =$(because $x > i$)$a^{i,i+1}(x-1) =$(because $x > i+1$ and therefore $x-1 > i$)$x-1$.

It follows that $a^{i,i+1}$ is indeed a model homomorphism.

With the notations above, let B be the directed colimit of $A;\widetilde{\beta}_{(S,F)};U$. Notice that $B_s = \{*\}$ because for any $i, j \in \mathbb{N}$ there exists $k \geq i, j$ such that $a^{i,k}(i) = k = a^{j,k}(j)$ and also notice that $B_{s\to s} = \{f_B, g_B\}$ where $f_B = f^i/_{\equiv}$, with $i \in \mathbb{N}$ and $g_B = g^i/_{\equiv}$, with $i \in \mathbb{N}$, because for any $i, j \in \mathbb{N}$ there exists no $k \geq i, j$ such that $f^k = g^k$. But $B_{apply_{s\to s}}(f_B, *) = * = B_{apply_{s\to s}}(g_B, *)$ which means B is not extensional.

We prove that there is no other model that is the colimit of this diagram, in two steps: first we show that $B/_{\Gamma_{(S,F)}}$ is not the colimit (which implies the colimit cannot interpret the sort s as a singleton) then we show that the colimit cannot have more than one element in the carrier of sort s.

Let us denote $C = B/_{\Gamma_{(S,F)}}$:

- $C_s = \{*\}$;
- $C_{s\to s} = \{1_*\}$, and $C_{apply_{s\to s}}(1_*,*) = *$;
- $C_f = C_g = 1_*$,

and let us denote for any $i \in \mathbb{N}$ $\mu_i : B^i \to C$, $\mu_i(x) = *$ and for any $x \in B^i_s$ and $\mu_i(f^i) = \mu_i(g^i) = 1_*$. It is easy to see that $(\{\mu_i\}_{i\in\mathbb{N}}, C)$ is a co-cone for $A;\widetilde{\beta}_{(S,F)}$.

Let D be the following $\phi(S,F)$-model:

- $D_s = \{*, y\}$;
- $D_{s\to s} = \{f_D, g_D\}$, and $D_{apply_{s\to s}}(f_D,*) = *$, $D_{apply_{s\to s}}(f_D,y) = y$, $D_{apply_{s\to s}}(g_D,*) = *$, $D_{apply_{s\to s}}(g_D,y) = *$.
- $D^f = f_D$, $D^g = g_D$

and let us define for any $i \in \mathbb{N}$ $\gamma_i : B^i \to D$, $\gamma_i(x) = *$, $\gamma_i(f^i) = f_D$, $\gamma_i(f^i) = g_D$.

We show that γ_i is model homomorphism. First notice that $\gamma_i(B^i_f) = D_f$ and $\gamma_i(B^i_g) = D_g$.

For any $x \in B^i_s$, $\gamma_i(B^i_{apply_{s\to s}}(f^i,x)) = *$ and $D_{apply_{s\to s}}(\gamma_i(f^i),\gamma_i(x)) = D_{apply_{s\to s}}(f_D,*) = *$.

For any $x \in B^i_s$, $\gamma_i(B^i_{apply_{s\to s}}(g^i,x)) = *$ and $D_{apply_{s\to s}}(\gamma_i(g^i),\gamma_i(x)) = D_{apply_{s\to s}}(g_D,*) = *$.

Therefore γ_i is a model homomorphism.

We show that $(\{\gamma_i\}_{i\in\mathbb{N}}, D)$ is a co-cone for $A;\widetilde{\beta}_{(S,F)}$.

For any $i < j \in \mathbb{N}$ and any $x \in B^i_s$, $\gamma_j(h^{i,j}(x)) = * = \gamma_i(x)$.

We also have that $\gamma_j(h^{i,j}(f^i)) = f_D = \gamma_i(f^i)$ and $\gamma_j(h^{i,j}(g^i)) = g_D = \gamma_i(g^i)$.

It follows that $(\{\gamma_i\}_{i\in\mathbb{N}}, D)$ is a co-cone for $A;\widetilde{\beta}_{(S,F)}$.

But then we cannot define a model homomorphism $\delta : C \to D$, because $D_f = \delta(C_f) \neq D_g = \delta(C_g)$ and $\delta(C_f) = \delta(C_g) = \delta(1_*)$.

We show that no model that interprets the sort s as a set with more than one element can be the directed colimit of $A;\widetilde{\beta}_{(S,F)}$.

Assume there exists such a model K such that $(\{\mu_i\}_{i\in\mathbb{N}}, K)$ is the directed colimit of $A;\widetilde{\beta}_{(S,F)}$.

Notice that:

- for any $i \in \mathbb{N}$ and any $x,y \in B^i_s$, $\mu_i(x) = \mu_i(y)$. For proving that, let $j > x,y$ and then $\mu_i(x) = \mu_j(h^{i,j}(x)) = \mu_j(j) = \mu_j(h^{i,j}(y)) = \mu_i(y)$.
- for any $i \leq j \in \mathbb{N}$, the functions μ_i and μ_j return the same constant value. Let $x > i,j$ and then $\mu_i(x) = \mu_j(h^{i,j}(x)) = \mu_j(x)$.

This implies that the set $K'_s = \{y \in K_s | y \neq \mu_i(x)\}$, where i and x are arbitrarily chosen, is not empty and independent of choice of i and x.

Let us consider the following model, L:

- $L_s = K_s \biguplus \{\bar{y} | y \in K'_s\}$;
- $L_{s \to s} = \{f_L, g_L\}$;
- for any $x \in K_s$, $L_{apply_{s \to s}}(f_L, x) = K_f(x)$ and $L_{apply_{s \to s}}(f_L, \bar{x}) = K_f(x)$;
- for any $x \in K_s$, $L_{apply_{s \to s}}(g_L, x) = K_g(x)$ and $L_{apply_{s \to s}}(g_L, \bar{x}) = K_g(x)$.

and we define for any $i \in \mathbb{N}$ and any $x \in B^i_s$, $\gamma_i(x) = \mu_i(x)$, $\gamma_i(f^i) = f_L$, $\gamma_i(g^i) = g_L$. It is easy to see that $(\{\gamma_i\}_{i \in \mathbb{N}}, L)$ is a co-cone of $A; \widetilde{\beta}_{(S,F)}$.

Let us consider $\delta_1, \delta_2 : K \to L$:

- $\delta_1(\mu_i(x)) = \mu_i(x)$ and $\delta_2(\mu_i(x)) = \mu_i(x)$;
- for any $x \in K'_s$, $\delta_1(x) = x$ and $\delta_2(x) = \bar{x}$;
- $\delta_i(f_K) = f_L$ and $\delta_i(g_K) = g_L$ for i either 1 or 2.

We have defined thus two model homomorphisms from K to L such that $\mu_i; \delta_i = \gamma_i$, which contradicts the assumption that K is the colimit.

■

Proposition 0.29 *If all $h^{i,j}$ are injective, then B is extensional.*

Proof: Let $f/_{\equiv}, g/_{\equiv} \in B_{s \to s'}$ such that $B_{apply_{s \to s}}(f/_{\equiv}, x/_{\equiv}) = B_{apply_{s \to s}}(g/_{\equiv}, x/_{\equiv})$ for any $x/_{\equiv} \in B_s$. We want to show that $f/_{\equiv} = g/_{\equiv}$.

Without loss of generality, we may assume that f and g are from the same algebra B^i. Let $y \in B^i_s$. We know by hypothesis that $B_{apply_{s \to s'}}(f/_{\equiv}, y/_{\equiv}) = B_{apply_{s \to s'}}(g/_{\equiv}, y/_{\equiv})$. By using the definition of the interpretation of $B_{apply_{s \to s'}}$ this means that $B^i_{apply_{s \to s'}}(f, y) \equiv B^i_{apply_{s \to s'}}(g, y)$ which with the definition of $\equiv$ further implies there exists $k \in |I|$ such that $h^{i,k}(B^i_{apply_{s \to s'}}(f, y)) = h^{i,k}(B^i_{apply_{s \to s'}}(g, y))$. By using the injectivity of $h^{i,k}$ we get $B^i_{apply_{s \to s'}}(f, y) = B^i_{apply_{s \to s'}}(g, y)$. Because y is arbitrarily chosen, we may apply the extensionality axiom for the algebra B^i to conclude that $f = g$. ■

Corollary 0.30 *HNK models have directed colimits of "injective" diagrams.*

0.6.7 Elementary diagrams

An institution $I = (\mathbb{S}ign, Sen, Mod, \models)$ has elementary diagrams [9] if and only if for each signature Σ and each Σ-model M, there exists a signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ (called the elementary extension of Σ via M), 'functorial' in Σ and M, and a set E_M of Σ_M-sentences(called the elementary diagram of the model M) such that $Mod(\Sigma_M, E_M)$ and the comma category $M/Mod(\Sigma)$ are naturally isomorphic i.e. the following diagram commutes by the isomorphism $i_{\Sigma,M}$ 'natural' in Σ and M.

$$\begin{array}{ccc} Mod(\Sigma_M, E_M) & \xrightarrow{i_{\Sigma,M}} & M/Mod(\Sigma) \\ & \searrow^{Mod(\iota_\Sigma(M))} & \downarrow{\scriptstyle forgetful} \\ & & Mod(\Sigma) \end{array}$$

Functoriality of ι means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each model homomorphism $h : M \to M' \upharpoonright_\varphi$ there exists a presentation morphism $i_\varphi(h) : (\Sigma_M, E_M) \to (\Sigma'_{M'}, E_{M'})$ such that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle \iota_\Sigma(M)}\downarrow & & \downarrow{\scriptstyle \iota_{\Sigma'}(M')} \\ \Sigma_M & \xrightarrow[i_\varphi(h)]{} & \Sigma'_{M'} \end{array}$$

commutes and $\iota_\varphi(h);\iota_{\varphi'}(h') = \iota_{\varphi;\varphi'}(h;h' \upharpoonright_\varphi)$ and $\iota_{1_\Sigma}(M) = 1_{\Sigma_M}$.

Naturality of i means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M' \upharpoonright_\varphi$ the following diagram commutes:

$$\begin{array}{ccc} Mod(\Sigma'_{M'}, E'_M) & \xrightarrow{i_{\Sigma',M'}} & M'/Mod(\Sigma') \\ {\scriptstyle Mod(i_\varphi(h))}\downarrow & & \downarrow{\scriptstyle h;(\ )\upharpoonright_\varphi} \\ Mod(\Sigma_M, E_M) & \xrightarrow[i_{\Sigma,M}]{} & M/Mod(\Sigma) \end{array}$$

Let $\Sigma = (S, F)$ be a HNK signature and let M be an (S, F)-model.

We define $\Sigma_M = (S, F_M)$, where $(F_M)_{\to s} = F_{\to s} \cup M_s$(i.e. the elements of M are added to the signature as constant symbols). Let $\iota_\sigma(M) : \Sigma \to \Sigma_M$ be the signature inclusion.

Let M_M be the Σ_M-model such that $(M_M) \upharpoonright_{\iota_\Sigma(M)} = M$ and that interprets all the elements of M as themselves: $(M_M)_m = m$, for any $m \in M$. Then we choose $E_M = \{t = t' | M_M \models t = t'\}$.

Proposition 0.31 *E_M is the elementary diagram of the model M.*

Proof:

We show that there exists an isomorphism of categories between $Mod(\Sigma_M, E_M)$ and $M/Mod(\Sigma)$. We denote $i_{\Sigma,M} : Mod(\Sigma_M, E_M) \to M/Mod(\Sigma)$ and $i^{-1}(\Sigma, M)$ the opposite functor.

The definition of $i_{\Sigma,M}$

- *on objects:*

  Let A be a Σ_M-model such that $A \models E_M$. We denote $i_{\Sigma,M}(A) = h_A : M \to A \upharpoonright_\Sigma$, which is defined by $(h_A)_s(m) = A_m$, for any $m \in M_s$.

Let us check that h_A is indeed a (S,F)-model homomorphism.

Let $\sigma \in F_s$. Notice that $M_M \models \sigma = M_\sigma$, because $(M_M)_\sigma = M_\sigma = (M_M)_{M_\sigma}$. Because $A \models E_M$, we have that $A \models \sigma = M_\sigma$, so $(A \upharpoonright_\Sigma)_\sigma = A_\sigma = A_{M_\sigma} = h_A(M_\sigma)$.

For any $f \in M_{s \to s'}$ and any $x \in M_s$, we must have that $h_A(f(x)) = h_A(f)(h_A(x))$. By using the definition of h_A, this is equivalent to $A_{f(x)} = A_f(A_x)$, which is true from the definition of the interpretation of Σ_M-terms.

- *on arrows:*

  Let $g' : A \to B$ be a Σ_M-model homomorphism. We denote $i_{\Sigma,M}(g') = g : (h_A : M \to A \upharpoonright_\Sigma) \to (h_B : M \to B \upharpoonright_\sigma)$. g must be a Σ-model homomorphism between $A \upharpoonright_\Sigma$ and $B \upharpoonright_\Sigma$, so we may choose $g = g' \upharpoonright_\Sigma$.

  We check that g is an arrow in the comma category, i.e. $h_A; g = h_B$.

  Let $m \in M_s$. Then $g(h_A(m)) = g(A_m) = g'(A_m) = B_m$ (by using the Σ_M-model homomorphism condition for g')$= h_B(m)$.

The definition of $i^{-1}_{\Sigma,M}$

- *on objects:*

  Let $h : M \to A$ be a (S,F)-model homomorphism. We denote $i^{-1}_{\Sigma,M}(h) = A_h$, which is the following (S,F_M)-model:

  - $(A_h) \upharpoonright_\Sigma = A$;
  - $(A_h)_m = h_s(m)$, for any $m \in M_s$.

  We check that $A_h \models E_M$.

  We show that $(A_h)_t = h((M_M)_t)$, for any term t, by structural induction on terms. If $t = \sigma$ with $\sigma \in F_s$, $(A_h)_\sigma = A_\sigma = h(M_\sigma) = h((M_M)_\sigma)$. If $t = m$ with $m \in M_s$, $(A_h)_m = h(m) = h((M_M)_m)$. If $t = t_1(t_2)$ and the induction hypothesis holds for t_1 and t_2, $(A_h)_{t_1(t_2)} = (A_h)t_1((A_h)_{t_2}) = h((M_M)_{t_1})(h((M_M)_{t_2})) = h((M_M)_{t_1}((M_M)_{t_2})) = h((M_M)_{t_1(t_2)})$.

  Therefore, if $M_M \models t = t'$, we have that $(M_M)_t = (M_M)_{t'}$, so $h((M_M)_t) = h((M_M)_{t'})$ which means $(A_h)_t = (A_h)_{t'}$, so $A_h \models t = t'$.

- *on arrows:*

  Let $g : (h_1 : M \to A) \to (h_2 : M \to B)$. We denote $i^{-1}(g) = g' : A_{h_1} \to B_{h_2}$ the function defined by $g'_s(a) = g_s(a)$ for any $a \in (A_{h_1})_s$.

  We check that g' is a Σ_M-model homomorphism.

  By using that g is a Σ-model homomorphism, we only have to show that $g'((A_{h_1})_m) = (B_{h_2})_m$, for any $m \in M$. But $g'((A_{h_1})_m) = g'(h_1(m)) = h_2(m) = (B_{h_2})_m$.

It is easy to see that $i_{\Sigma,M}$ is an isomorphism of categories.

"Functoriality" of ι

Let $\varphi : (S,F) \to (S',F')$ be a signature morphism. Let M' be a (S',F')-model and $f : M \to M' \upharpoonright_\varphi$ a (S,F)-model homomorphism.

We define $i_\varphi(f) : ((S,F_M),E_M) \to ((S',F'_{M''},E_{M'}))$ such that the following diagram commutes

$$\begin{array}{ccc} (S,F) & \xrightarrow{\ \varphi\ } & (S',F') \\ {\scriptstyle \iota_\Sigma(M)}\downarrow & & \downarrow{\scriptstyle \iota_{\Sigma'}(M')} \\ (S,F_M) & \xrightarrow[i_\varphi(f)]{} & (S',F'_{M'}) \end{array}$$

For each type $s \in \overrightarrow{S}$, let $i_\varphi(f)(s) = \varphi(s)$. For each constant symbol $\sigma \in F_s$, we define $i_\varphi(f)(\sigma) = \varphi(\sigma)$ and for each $m \in M_s$, $i_\varphi(f)(m) = f(m)$. Notice that $f(m) \in (M' \upharpoonright_\varphi)_s = M'_\varphi(s)$, so $i_\varphi(f)(m)$ is a constant symbol of type $\varphi(s) = i_\varphi(f)(s)$ and therefore $i_\varphi(f)$ is indeed a HNK-signature morphism.

We only have to check that $i_\varphi(f)$ is a presentation morphism. We will prove that $M_M \models t = t'$ implies $M'_{M'} \models Sen^{HNK}(i_\varphi(f))(t = t')$. According to the satisfaction condition for HNK, this is equivalent to $M_M \models t = t'$ implies $N \models t = t'$, where we denote $N = (M'_{M'}) \upharpoonright_{i_\varphi(f)}$. We notice that $N_t = f((M_M)_t)$, for any term t, by structural induction on terms. If $\sigma \in F_{\to s}$, $N_\sigma = (M'_{M'})_{i_\varphi(f)(\sigma)} = (M'_{M'})_{\varphi(\sigma)} = M'_{\varphi(\sigma)} = (M' \upharpoonright_\varphi)_\sigma = f(M_\sigma) = f((M_M)_\sigma)$. If $m \in M_s$, $N_m = (M'_{M'})_{i_\varphi(f)(m)} = (M'_{M'})_{f(m)} = f(m) = f((M_M)_m)$. If $t = t_1(t_2)$ and the inductive hypothesis holds for t_1 and t_2, $N_{t(t')} = N_t(N'_t) = f((M_M)_t)(f((M_M)'_t)) = f((M_M)_t((M_M)'_t)) = f((M_M)_{t(t')})$.

Therefore, if $M_M \models t = t'$, we have that $(M_M)_t = (M_M)_{t'}$, so $f((M_M)_t) = f((M_M)_{t'})$ which means $N_t = N_{t'}$ and then $N \models t = t'$.

"Naturality" of i

Let $\varphi : (S,F) \to (S',F')$ be a signature morphism. Let M' be a (S',F')-model and $f : M \to M' \upharpoonright_\varphi$ a (S,F)-model homomorphism.

We must check that the following diagram commutes:

$$\begin{array}{ccc} Mod((S',F'_{M'}),E'_M) & \xrightarrow{\ i_{\Sigma',M'}\ } & M'/Mod(S',F') \\ {\scriptstyle Mod(i_\varphi(f))}\downarrow & & \downarrow{\scriptstyle f;(\ )\upharpoonright_\varphi} \\ Mod((S,F_M),E_M) & \xrightarrow[i_{\Sigma,M}]{} & M/Mod(S,F) \end{array}$$

Let A' be a (Σ_M,E_M)-model. We have to check that $f;(h_A) \upharpoonright_\Sigma = h_{(A\upharpoonright_{i_\varphi(f)})}$.

Notice that $f;(h_A) \upharpoonright_\Sigma: M \to (A' \upharpoonright_{\iota_{\Sigma'}(M')}) \upharpoonright_\varphi$ and $h_{(A\upharpoonright_{i_\varphi(f)})} : M \to (A' \upharpoonright_{i_\varphi(f)}) \upharpoonright_{\iota_\Sigma(M)}$. Because of the functoriality condition, we have that the morphisms have the same domain and the same codomain.

Let $m \in M_s$. $(f;(h_A) \upharpoonright_\Sigma)(m) = (h_A) \upharpoonright_\Sigma (f(m)) = A_{f(m)}$. $h_{(A\upharpoonright_{i_\varphi(f)})}(m) = (A \upharpoonright_{i_\varphi(f)})_m = A_{i_\varphi(f)(m)} = A_{f(m)}$.

Let $g : A \to B$ be a (Σ_M,E_M)-model homomorphism.

On one hand, $i_{\Sigma,M}(g) = g \restriction_{\iota_{\Sigma'}(M')}$ and $f/Mod(\varphi)(g \restriction_{\iota_{\Sigma'}(M')}) = g \restriction_{\iota_{\Sigma'}(M')} \restriction_{\varphi}$.

On the other hand, $i_{\Sigma,M}(g \restriction_{i_\varphi(f)}) = g \restriction_{i_\varphi(f)} \restriction_{\iota_\Sigma(M)}$.

By using the functoriality condition we obtain that they are the same.

■

0.6.8 Basic sentences

A set of sentences $E \in Sen(\Sigma)$ is called *basic* [8] if there exists a Σ-model M_E such that, for all Σ-models M, $M \models E$ iff there exists a homomorphism $M_E \to M$. Basic sentences tend to be the starting building blocks for sentences in concrete institutions and are usually (with a supplementary requirement that the homomorphism is unique) the best approximation for atomic formulas.

Counterexample 5 *In HNK, not all atomic sentences are basic.*

Let us consider the following HNK signature:

- $S = \{s, s'\}$;
- $F = \{f : s \to s;\ g, h : (s \to s) \to s'\}$.

We show that the sentence $g(f) = h(f)$ is not basic.
Let us consider the following HNK-model:

- $M_s = \emptyset$;
- $M_{s'} = \{a\}$;
- $M_{s \to s} = \{1_\emptyset\}$
- $M_{(s \to s) \to s'} = \{F\}$, *where* $F(1_\emptyset) = a$;
- $M_f = 1_\emptyset$, $M_g = M_h = F$.

Notice that $M \models g(f) = h(f)$ and also that there exist models N such that $N \models g(f) = h(f)$ and no model homomorphism from M to N can be defined (any models N with $N_g \neq N_h$).

Assume that $g(f) = h(f)$ is basic and let M_e be the model for which $N \models g(f) = h(f) \iff$ there exists a model homomorphism $\varphi_N : M_e \to N$.

Because $M \models g(f) = h(f)$, there exists a model homomorphism $\varphi_M : M_e \to M$. Since the codomain of $(\varphi_M)_s$ is empty, $(M_e)_s$ must be empty too. It follows that $(M_e)_{s \to s}$ is $\{1_\emptyset\}$ (it cannot be the empty set because f has an interpretation in M_e).

The first case further implies $(M_e)_f = 1_\emptyset$. We know that $M_e \models e$, so $(M_e)_g((M_e)_f) = (M_e)_h((M_e)_f)$. It follows that $(M_e)_g$ and $(M_e)_h$ are equal, since they return the same value when applied to each element in their domain, and this cannot be true for M_e.

0.6.9 Sentences preserved by ultraproducts. Compactness

Definition 15 *Given a class of filters $\mathcal{F}$, a Σ-sentence e is*

- *preserved by $\mathcal{F}$-filtered factors if $\Pi_F A_i \models e \implies \{i \in I | A_i \models e\} \in F$, for any $F \in \mathcal{F}$*
- *preserved by $\mathcal{F}$-filtered products if $\{i \in I | A_i \models e\} \in F \implies \Pi_F A_i \models e$, for any $F \in \mathcal{F}$*

If $\mathcal{F}$ is the class of all ultrafilters, we say that the sentence e is preserved by ultrafactors, respectively ultraproducts.

Definition 16 *A sentence is Łoś when it is preserved by all ultrafactors and all ultraproducts.*

Definition 17 *An institution is Łoś if and only if it has all ultraproducts of models and all its sentences are Łoś sentences.*

Definition 18 *[6] An institution comorphism $(\phi, \alpha, \beta) : I \to I'$ is*

- *liberal if β_Σ has a left adjoint for any I-signature Σ;*
- *persistently liberal when the adjunctions between categories of models are persistent.*

Proposition 0.32 *[6] For each persistently liberal comorphism $(\phi, \alpha, \beta) : I \to I'$, if I' is a Łoś institution then I is also a Łoś institution.*

Remark 7 *$FOEQL$ is a Łoś institution.*

Remark 8 *$FOEQL^p$ is a Łoś institution.*

Proof:

First let us notice that $Mod(\Sigma, E)$ has all ultraproducts which are precisely those in $Mod(\Sigma)$, because E contains sentences which are preserved by ultraproducts. All sentences in $FOEQL^p$ are therefore Łoś sentences, because they are Łoś sentences in $FOEQL$ and the satisfaction relation in $FOEQL^p$ coincides with the one in $FOEQL$.

Remark 9 *The institution comorphism $(\phi, \alpha, \beta) : HNK \to FOEQL^p$ defined in section 0.4 is persistently liberal.*

Corollary 0.33 *HNK is a Łoś institution.*

Proposition 0.34 *[6]*

Each Łoś institution is (m-)compact.

Corollary 0.35 *HNK is compact.*

0.6.10 Craig interpolation

Definition 19 *In an institution $I = (\mathbb{S}ign, Sen, Mod, \models)$ let us consider the following commuting square of signature morphisms:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

If for any sets of sentences $E_1 \subseteq Sen(\sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\theta_1(E_1) \models \theta_2(E_2)$ there exists an interpolant $E \subseteq Sen(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models E_2$, the square is called Craig interpolation square.

Definition 20 *Let $I = (\mathbb{S}ign, Sen, Mod, \models)$ be an institution and let us consider two classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms. We say that the institution has the Craig $(\mathcal{L}, \mathcal{R})$-interpolation property if each pushout square of signature morphisms*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

with $\varphi_1 \in \mathcal{L}$ and $\varphi_2 \in \mathcal{R}$ is a Craig interpolation square.

Proposition 0.36 *[6]*

In any institution with universal $\mathcal{R}$-quantification for a class $\mathcal{R}$ of signature morphisms, any weak amalgamation square

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

for which φ_2 is in $\mathcal{R}$ is a Craig interpolation square.

Corollary 0.37 *HNK has Craig $(\mathbb{S}ign, \mathbb{E}xt)$-interpolation, where Ext is the class of all signature extensions with constants.*

Proof:

We have to establish that *HNK* admits universal $\mathcal{R}$-quantification.

Let $\varphi : (S, F) \to (S, F \cup F')$ be a signature morphism in $\mathcal{R}$ and let ρ' be a $(S, F \cup F')$-sentence. We have to prove that $(\forall \varphi)\rho'$ is semantically equivalent to a Σ-sentence, the problem being when φ is an extension of Σ with an infinite number of symbols.

Because ρ' is finitary, there exists a sub-signature (S_0, F_0) of $(S, F \cup F')$ such that (S_0, F_0) has a finite number of sorts and constant symbols and ρ' is a (S_0, F_0)-sentence.

Then the square below

$$\begin{array}{ccc} (S,F) \cap (S_0,F_0) & \xrightarrow{\varphi_0} & (S_0,F_0) \\ \downarrow & & \downarrow \\ (S,F) & \xrightarrow{\varphi} & (S,F \cup F') \end{array}$$

is a weak amalgamation square.This is true because $(S \cup S_0, F \cup F_0)$ is the push-out of the span of signature extensions $(S,F) \leftarrow (S,F) \cap (S_0,F_0) \rightarrow (S_0,F_0)$, the institution has weak model amalgamation and the signature inclusion $(S_0,F_0) \rightarrow (S,F \cup F')$ is conservative. It is then easy to see that $(\forall\varphi)\rho'$ is semantically equivalent to $(\forall\varphi_0)\rho'$.

■

Borrowing interpolation

Definition 21 *[6]*

In an institution $I = (\mathbb{S}ign, Sen, Mod, \models)$ let us consider the following commuting square of signature morphisms:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

If for any sets of sentences $E_1 \subseteq Sen(\sigma_1)$ and $E_2, \Gamma_2 \subseteq Sen(\Sigma_2)$ such that $\theta_1(E_1) \cup \theta_2(\Gamma_2) \models \theta_2(E_2)$ there exists an interpolant $E \subseteq Sen(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\Gamma_2 \cup \varphi_2(E) \models E_2$, the square is called Craig-Robinson interpolation square.

Proposition 0.38 *[6]*

In any institution that has implications and is compact, a commuting square of signature morphisms is a Craig-Robinson Interpolation square if and only if is a Craig Interpolation square.

Proposition 0.39 *[6]*

Let $I = (\mathbb{S}ign, Sen, Mod, \models)$ be any institution. For each class $\mathcal{S} \subseteq \mathbb{S}ign$ of signature morphisms, let $\mathcal{S}^p$ be the class of presentation morphisms $\varphi : (\Sigma, E) \rightarrow (\Sigma', E')$ such that $\varphi \in \mathcal{S}$. Then the institution $(\mathbb{P}res, Sen^p, Mod^p, \models)$ of the presentations of I has the Craig-Robinson ($\mathcal{L}^p$, $\mathcal{R}^p$)-interpolation if I has the Craig − Robinson $(\mathcal{L}, \mathcal{R})$-interpolation.

Remark 10 *By using that $FOEQL$ has Craig $(\mathbb{I}nj, \mathbb{S}ign)$-interpolation, where $\mathbb{I}nj$ is the class of signature morphisms that are injective on sorts and that both $FOEQL$ and $FOEQL^p$ are compact and have implication we conclude that $FOEQL^p$ also has Craig $(\mathbb{I}nj, \mathbb{S}ign)$-interpolation.*

Interpolation property for comorphisms

For a fixed class $\mathcal{S} \subseteq \mathbb{S}ig$ of signature morphisms, we say that an institution comorphism $(\phi, \alpha, \beta) : I \to I'$:

- has the Craig $\mathcal{S}$-left interpolation property when for each signature morphism $\varphi_1 : \Sigma \to \Sigma_1$ in $\mathcal{S}$, each set E_1 of Σ_1 sentences and each set E_2 of $\phi(\Sigma_1)$-sentences such that $\alpha_{\Sigma_1}(E_1) \models' \phi(\varphi_1)(E_2)$, there exists a set of Σ-sentences E such that $E_1 \models \varphi_1(E)$ and $\alpha_\Sigma(E) \models' E_2$,
- has the Craig $\mathcal{S}$-right interpolation property when for each signature morphism $\varphi_2 : \Sigma \to \Sigma_2$ in $\mathcal{S}$, each set E_1 of $\phi(\Sigma)$-sentences and each set E_2 of Σ_2-sentences such that $\phi(\varphi_2)(E_1) \models' \alpha_{\Sigma_2}(E_2)$, there exists a set of Σ-sentences E such that $E_1 \models' \alpha_\Sigma(E)$ and $\phi_2(E) \models E_2$.

Lemma 0.40 *Let*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

be a Craig interpolation square and let $\chi : \Sigma' \to \Sigma''$ *be a conservative signature morphism. Then the square*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1;\chi} \\ \Sigma_2 & \xrightarrow{\theta_2;\chi} & \Sigma'' \end{array}$$

is still a Craig interpolation square.

Proof:

Let $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $(\theta_1;\chi)(E_1) \models (\theta_2;\chi)(E_2)$. We show that $\theta_1(E_1) \models \theta_2(E_2)$ and then we obtain the interpolant E by applying the Craig interpolation property for the square

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

Let $M' \in |Mod(\Sigma')|$ such that $M' \models \theta_1(E_1)$. By using the conservativity of χ we get a χ-expansion $M'' \in |Mod(\Sigma'')|$ of M' and with the satisfaction condition we obtain $M'' \models (\theta_1;\chi)(E_1)$. It follows that $M'' \models (\theta_2;\chi)(E_2)$ so by using the satisfaction condition again $M' \models \theta_2(E_2)$.

■

Definition 22 *Let* $(\phi, \alpha, \beta) : I \to I'$ *be an institution comorphism. We say that* (ϕ, α, β) *conservatively approximates signature pushouts if any signature pushout in* I

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

is mapped via ϕ to a commutative square

$$\begin{array}{ccc} \phi(\Sigma) & \xrightarrow{\phi(\varphi_1)} & \phi(\Sigma_1) \\ \downarrow{\scriptstyle \phi(\varphi_2)} & & \downarrow{\scriptstyle \phi(\theta_1)} \\ \phi(\Sigma_2) & \xrightarrow{\phi(\theta_2)} & \phi(\Sigma') \end{array}$$

such that the unique signature morphism from the pushout of the span $\phi(\Sigma_2) \stackrel{\phi(\varphi_2)}{\longleftarrow} \phi(\Sigma) \stackrel{\phi(\varphi_1)}{\longrightarrow} \phi(\Sigma_1)$ *to* $\phi(\Sigma')$ *is conservative.*

Remark 11 *Each institution comorphism (ϕ, α, β) such that ϕ preserves pushouts conservatively approximates pushouts (because each identity in $\mathbb{S}ign'$ is conservative).*

Definition 23 *[6]*

An institution comorphism (ϕ, α, β) is conservative when for each signature Σ and each Σ-model M there exists a $\phi(\Sigma)$-model M' such that $\beta_\Sigma(M') = M$.

Theorem 0.41 *Let $(\phi, \alpha, \beta) : I \to I'$ be a conservative institution comorphism that conservatively approximates signature pushouts and let $\mathcal{L}$, $\mathcal{R} \subseteq \mathbb{S}ig$ be classes of signature morphisms such that I' has the Craig $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-interpolation property.*

If (ϕ, α, β) has the Craig $\mathcal{L}$-left or $\mathcal{R}$-right interpolation property, then I has Craig $(\mathcal{L}, \mathcal{R})$-interpolation.

Proof:

Let

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \end{array}$$

be a pushout of signature morphisms such that $\phi_1 \in \mathcal{L}$ and $\phi_2 \in \mathcal{R}$ and let $E_1 \subseteq Sen(\Sigma_1)$, $E_2 \subseteq Sen(\Sigma_2)$ such that $\theta_1(E_1) \models \theta_2(E_2)$.

The latter relation leads to $\alpha_{\Sigma'}(\theta_1(E_1)) \models \alpha_{\Sigma'}(\theta_2(E_2))$ which by the naturality of α further leads to the interpolation problem $\phi(\theta_1)(\alpha_{\Sigma_1}(E_1)) \models \phi(\theta_2)(\alpha_{\Sigma_2}(E_2))$ for the following commutative square of signature morphisms in I'

$$\begin{array}{ccc} \phi(\Sigma) & \xrightarrow{\phi(\varphi_1)} & \phi(\Sigma_1) \\ \downarrow{\scriptstyle \phi(\varphi_2)} & & \downarrow{\scriptstyle \phi(\theta_1)} \\ \phi(\Sigma_2) & \xrightarrow{\phi(\theta_2)} & \phi(\Sigma') \end{array}$$

By using the Craig $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-interpolation property of I' and lemma 0.40 we get that the square is a Craig $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-interpolation square, so we find $E_0 \subseteq Sen'(\phi(\Sigma))$ such that $\alpha_{\Sigma_1}(E_1) \models \phi(\varphi_2)(E_0) \models \alpha_{\Sigma_2}(E_2)$.

Let us assume that the institution comorphism has Craig $\mathcal{L}$-left interpolation property. Then we can find $E \subseteq Sen(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\alpha_\Sigma(E) \models E_0$. By applying $\phi(\varphi_2)$ we get that $\phi(\varphi_2)(\alpha_\Sigma(E)) \models$

$\phi(\varphi_2)(E_0)$. By the naturality of α we get $\alpha_{\Sigma_2}(\varphi_2(E)) \models \phi(\varphi_2)(E_0) \models \alpha_{\Sigma_2}(E_2)$ and by using the conservativity of the institution comorphism we get $\varphi_2(E) \models E_2$.

The case when the institution comorphism has $\mathcal{R}$-right interpolation is similar.

■

Borrowing interpolation along the institution comorphism from HNK to $FOEQL^p$.

Remark 12 *Let $\phi : \Sigma \to \Sigma'$ be a HNK-signature morphism such that ϕ^{sort} is injective. Then ϕ^{type} is injective too.*

Let us denote $\mathbb{I}nj$ the class of HNK signature morphisms that are injective on sorts.

Lemma 0.42 *The institution comorphism $(\phi,\alpha,\beta) : HNK \to FOEQL^p$ has Craig $(\mathbb{I}nj, \mathbb{S}ig)$-interpolation property.*

Proof:

Let $\varphi_1 : (S,F) \to (S_1,F_1)$ be a signature morphism from $\mathbb{I}nj$ and let $E_1 \subseteq Sen(S_1,F_1)$, $E_2 \subseteq Sen(\phi(S,F))$ such that $\alpha_{(S_1,F_1)}(E_1) \models \phi(\varphi)(E_2)$.

Let us consider the following pushout of signature morphisms:

$$\begin{array}{ccc} \phi(S,F) & \xrightarrow{\phi(\varphi_1)} & \phi(S_1,F_1) \\ \downarrow 1_{\phi(S,F)} & & \downarrow 1_{(\phi(S_1,F_1))} \\ \phi(S,F) & \xrightarrow{\phi(\varphi_1)} & \phi(S_1,F_1) \end{array}$$

Because $FOEQL^p$ has Craig $(\mathbb{I}nj, \mathbb{S}ign)$-interpolation, we obtain $E_0 \subseteq Sen(\phi(S,F))$ such that $\alpha_{(S_1,F_1)}(E_1) \models \phi(\varphi)(E_0)$ and $E_0 \models E_2$. We obtain thus $E = \alpha^{-1}_{(S,F)}(E_0)$.

■

Remark 13 *Because $\beta_{(S,F)}$ is an equivalence of categories for any HNK signature (S,F), the institution comorphism $HNK \to FOEQL^p$ is conservative.*

Corollary 0.43 *HNK has Craig $(\mathbb{I}nj, \mathbb{S}ign)$-interpolation.*

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